Introductory Lectures on Theoretical Physics

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1 Vibrational modes and one dimensional lattice

1.1 Lagrangian and Hamiltonian

Let us consider the elastic vibrations of a cubic cristal. The solution is simple when the waves describing the vibrations propagate in the x, y or z direction because entire planes of atoms oscillate.

For example for a two dimensional cristal, let us consider a wave propagating as $\hat{\mathbf{k}} = \hat{\mathbf{x}}$. Then one has a longitudinal wave when

$$\mathbf{u}_j = \hat{\mathbf{x}} \exp\left[i(kx_j - \omega t)\right] \tag{1.1}$$

where u_j is the displacement of the j atom, $x_j = ja$ being a the lattice size. One has a transverse wave when

$$\mathbf{u}_j = \hat{\mathbf{y}} \exp\left[i(kx_j - \omega t)\right] \tag{1.2}$$

In the first case the displacement of the j atom u_j is in the x direction while in the second case is in the y direction. In both cases the problem is reduced to a one dimensional case. Therefore for simplicity we will consider a one dimensional lattice.

Let us consider a one dimensional chain, containing N atoms with spacing a, bound by an elastic force with elastic constant C. Let m be the mass of the atoms and $x_i = ia$, $i = 1, \ldots N$, the rest (equilibrium) position of the atoms and x'_i the position at the time t. Then the displacement with respect to the equilibrium position is

$$u_i = x'_i - x_i \tag{1.3}$$

For the study we will assume boundary periodic conditions

$$u_{N+1} = u_1 \tag{1.4}$$

The potential of the chain is given by

$$V = \frac{1}{2}C[(u_1 - u_2)^2 + (u_2 - u_3)^2 + \cdots] = \frac{1}{2}C\sum_{i=1}^{N}(u_i - u_{i+1})^2 \tag{1.5}$$

the kinetic energy

$$\frac{m}{2}\sum_{i=1}^{N}\dot{u}_i^2 \tag{1.6}$$

Then the Lagrangian for the one-dimensional lattice is given by

$$L = \frac{m}{2}\sum_{i=1}^{N}\dot{u}_i^2 - \frac{1}{2}C\sum_{i=1}^{N}(u_i - u_{i+1})^2 \tag{1.7}$$

The Lagrangian describes the small oscillations of the atoms with respect to the equilibrium. Higher order terms, cubic or quartic in u_i, could be introduced. For simplicity we limit ourselves to (1.7).

Let us now consider the Euler equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_i} = \frac{\partial L}{\partial u_i}, \quad i = 1 \ldots N \tag{1.8}$$

or

$$\begin{aligned} m\ddot{u}_i &= \frac{\partial V}{\partial u_i} = -\frac{C}{2}2\sum_j (u_j - u_{j+1})(\delta_{i,j} - \delta_{i,j+1}) \\ &= -C(u_i - u_{i+1}) + C(u_{i-1} - u_i) \\ &= -C(2u_i - u_{i-1} - u_{i+1}) \\ &= -\sum_j V_{ij}u_j, \quad i,j = 1 \ldots N \end{aligned} \tag{1.9}$$

with

$$V_{ij} = C(2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}), \quad i,j = 1 \ldots N \tag{1.10}$$

This is a finite difference differential equation, which can be solved by a discrete Fourier transform,

$$u_j(t) = A\exp(i\chi j)\exp(-i\omega_\chi t) + cc, \quad j = 1 \ldots N \tag{1.11}$$

By substituting (1.11) in eq.(1.9), we get

$$\begin{aligned} m\left[A\exp(i\chi j)\exp(-i\omega_\chi t) + cc\right](-\omega_\chi^2) &= -C[2A\exp(i\chi j)\exp(-i\omega_\chi t) + cc] \\ &+ C[A\exp(i\chi(j-1))\exp(-i\omega_\chi t) + cc] \\ &+ C[A\exp(i\chi(j+1))\exp(-i\omega_\chi t) + cc] \end{aligned} \tag{1.12}$$

and therefore

$$\omega_\chi^2 = 2\frac{C}{m}(1-\cos\chi) = 4\frac{C}{m}\sin^2\frac{\chi}{2} \tag{1.13}$$

or assuming positive frequency

$$\omega_\chi = 2\sqrt{\frac{C}{m}}|\sin\frac{\chi}{2}| \tag{1.14}$$

By imposing the condition (1.4) we get

$$\exp\left[i\chi(N+1)\right] = \exp\left(i\chi\right) \tag{1.15}$$

implying

$$\exp\left(i\chi N\right) = 1, \quad \chi = \frac{2\pi n}{N} \tag{1.16}$$

with $-N/2+1 \leq n \leq N/2$ for even N and $-(N-1)/2 \leq n \leq (N-1)/2$ for odd N. In conclusion one has N proper modes. Since χ is defined modulus 2π, n is defined modulus N. In fact if $\chi \to \chi + 2m\pi$, then

$$n = \frac{\chi N}{2\pi} \to \frac{\chi N}{2\pi} + \frac{2m\pi N}{2\pi} = n + mN \tag{1.17}$$

We can rewrite the solution as

$$\begin{aligned} u_j(t) &= A\exp\left(i\chi j - i\omega_\chi t\right) + cc \\ &= A\exp\left(ikx_j - i\omega_\chi t\right) + cc \\ &= A\exp\left(ikx_j - i\omega_n t\right) + cc \end{aligned} \tag{1.18}$$

with

$$k = \frac{\chi}{a} = \frac{2\pi n}{Na} \tag{1.19}$$

and

$$\omega_n = 2\sqrt{\frac{C}{m}}|\sin\frac{\pi n}{N}| \tag{1.20}$$

Note that all atoms oscillate with the same frequency ω_n. The group velocity

$$v_g = \frac{d\omega}{dk} = \sqrt{\frac{C}{m}}a\cos\frac{\pi n}{N} \tag{1.21}$$

Furthermore the condition (1.19) gives

$$n\lambda = n\frac{2\pi}{k} = Na \tag{1.22}$$

or the total length must contain an integer number of wave lengths.

1.2 Expansion in eigenmodes and quantization

Since the $u_i^{(n)}$ are a set of orthonormal and complete functions, the general solution can be written in terms of the normal modes, defined as

$$u_j(t) \sim u_j^{(n)} \exp(-i\omega_\chi t) + cc \tag{1.23}$$

$$u_j^{(n)} = \frac{1}{\sqrt{N}} \exp(i\frac{2\pi n}{N} j) \tag{1.24}$$

Let us show that (1.24) are a set of orthonormal functions, or

$$\sum_j (u_j^{(n)})^* u_j^{(m)} = \delta_{n,m} \tag{1.25}$$

and satisfy the completeness relation

$$\sum_n (u_i^{(n)})(u_j^{(n)})^* = \delta_{i,j} \tag{1.26}$$

For $i = j$ or $n = m$ the properties are obvious while the orthogonality (completeness) condition follows from the fact that by inserting eq.(1.23) in the equations of motion (1.9), we get

$$V_{ij} u_j^{(n)} = m\omega_n^2 u_i^{(n)} = m\omega_n^2 \delta_{ij} u_j^{(n)} \tag{1.27}$$

which means that the normal modes $u_i^{(n)}$ are eigenvectors of the matrix V_{ij} with eigenvalues $m\omega_n^2$. Eigenvectors corresponding to distinct eigenvalues are orthogonal.

The general solution is therefore

$$u_j(t) = \sum_n A_n u_j^{(n)} \exp(-i\omega_n t) + c.c., \tag{1.28}$$

with A_n complex or with a different normalization and notation ($\omega_n \to \omega_k$)

$$u_j(t) = \sum_k \sqrt{\frac{\hbar}{2m\omega_k}} a_k u_j^{(k)} \exp(-i\omega_k t) + c.c. \tag{1.29}$$

where we recall that $\omega_k = 2\sqrt{C/m}\sin(ka/2)$ with $k = 2\pi n/Na$. The momentum is given by

$$p_j(t) = \frac{\partial L}{\partial \dot{u}_j} = m\dot{u}_j(t) = m\sum_k \sqrt{\frac{\hbar}{2m\omega_k}} a_k u_j^{(k)}(-i\omega_k)\exp(-i\omega_k t) + c.c. \tag{1.30}$$

The Hamiltonian is

$$H = -L + \sum_j \dot{u}_j p_j = T + V \tag{1.31}$$

Let us now invert the relations in order to obtain a_k in terms of the coordinates and momenta. Let us consider

$$u_j(0) \equiv q_j(0) = \sum_k \sqrt{\frac{\hbar}{2m\omega_k}}(a_k u_j^{(k)} + a_k{}^*(u_j^{(k)})^*) \tag{1.32}$$

$$p_j(0) = \sum_k \frac{\sqrt{m\omega_k\hbar}}{i\sqrt{2}}(a_k u_j^{(k)} - a_k{}^*(u_j^{(k)})^*) \tag{1.33}$$

By multiplying by $(u_j^{(k')})^*$ and summing over j we get

$$\sum_j (u_j^{(k')})^* q_j(0) = \sqrt{\frac{\hbar}{2m\omega_{k'}}}(a_{k'} + a^*_{-k'}) \tag{1.34}$$

$$\sum_j (u_j^{(k')})^* p_j(0) = \frac{1}{i}\sqrt{\frac{m\omega_{k'}\hbar}{2}}(a_{k'} - a^*_{-k'}) \tag{1.35}$$

from which

$$a_k = \sum_j (u_j^{(k)})^*(\sqrt{\frac{m\omega_k}{2\hbar}} q_j(0) + i\sqrt{\frac{1}{2m\omega_k\hbar}} p_j(0)) \tag{1.36}$$

We can now quantize the theory by requiring the standard commutation relations

$$[q_i, p_j] = i\hbar\delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0 \tag{1.37}$$

By using eq.(1.36) and eq. (1.37), we get

$$[a_k, a^\dagger_{k'}] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a^\dagger_k, a^\dagger_{k'}] = 0 \tag{1.38}$$

Since the Hamiltonian does not depend explicitly on t we can quantize at $t = 0$

$$H(t=0) = \frac{1}{2m}\sum_j p_j^2(0) + \frac{1}{2}\sum_{l,j} V_{lj} q_l(0) q_j(0) \tag{1.39}$$

where

$$V_{lj} = \frac{\partial^2 V}{\partial u_l \partial u_j} = C(2\delta_{l,j} - \delta_{l,j-1} - \delta_{l,j+1}) \tag{1.40}$$

By substituting in the Hamiltonian $p_j(0)$ and $q_j(0)$ we get

$$H = \frac{1}{2}\sum_k \hbar\omega_k (a_k^\dagger a_k + a_k a_k^\dagger) \tag{1.41}$$

In fact we have

$$\begin{aligned}
\frac{1}{2m}\sum_j p_j^2(0) &= \frac{\hbar}{2m}\sum_j\sum_k\sum_{k'} \frac{\sqrt{m\omega_k}}{i\sqrt{2}}\frac{\sqrt{m\omega_{k'}}}{i\sqrt{2}} \\
&\quad (a_k u_j^{(k)} - a_k{}^\dagger (u_j^{(k)})^*)(a_{k'} u_j^{(k')} - a_{k'}{}^\dagger (u_j^{(k')})^*) \\
&= -\frac{\hbar}{4}\sum_k\sum_{k'} \sqrt{\omega_k}\sqrt{\omega_{k'}}[\delta_{k,-k'}(a_k a_{k'} + a_k^\dagger a_{k'}^\dagger) \\
&\quad -\delta_{k,k'}(a_k a_{k'}^\dagger + a_k^\dagger a_{k'})] \\
&= -\frac{\hbar}{4}\sum_k \omega_k (a_k a_{-k} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger)
\end{aligned} \tag{1.42}$$

Furthermore we have

$$\begin{aligned}
\frac{1}{2}\sum_{l,j} V_{lj} q_l(0) q_j(0) &= \frac{\hbar}{2}\sum_{l,j} V_{lj} \sum_k\sum_{k'} \frac{1}{\sqrt{2m\omega_k}}\frac{1}{\sqrt{2m\omega_{k'}}} \\
&\quad (a_k u_l^{(k)} + a_k{}^\dagger (u_l^{(k)})^*)(a_{k'} u_j^{(k')} + a_{k'}{}^\dagger (u_j^{(k')})^*) \\
&= \frac{\hbar}{2}\sum_k\sum_{k'} \frac{1}{\sqrt{2m\omega_k}}\frac{1}{\sqrt{2m\omega_{k'}}} m\omega_k^2 \\
&\quad [\delta_{k,-k'}(a_k a_{k'} + a_k^\dagger a_{k'}^\dagger) + \delta_{k,k'}(a_k a_{k'}^\dagger + a_k^\dagger a_{k'})] \\
&= \frac{\hbar}{4}\sum_k \omega_k (a_k a_{-k} + a_k a_k^\dagger + a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger)
\end{aligned} \tag{1.43}$$

where use has been made eqs. (1.27) and of the orthonormality of the $u_j^{(k)}$. Then the Hamiltonian is given by

$$H = \frac{\hbar}{2}\sum_k \omega_k(a_k^\dagger a_k + a_k a_k^\dagger) = \sum_k \hbar\omega_k(a_k^\dagger a_k + \frac{1}{2}) \tag{1.44}$$

The Hamiltonian is then the sum of N armonic oscillator Hamiltonians. The Hilbert space is built starting from the fundamental state (in quantum field theory this state is called the vacuum) $|0>=|0>_{k_{min}}\cdots|0>_{k_{max}}$ such that

$$a_k|0>=0,\ \ \forall k \tag{1.45}$$

Remember $k = 2\pi n/Na$ with (for example for even N)

$$k_{min} = \frac{2\pi(-\frac{N}{2}+1)}{Na},\ \ k_{max} = \frac{2\pi\frac{N}{2}}{Na} \tag{1.46}$$

The state

$$a_k^\dagger|0> \tag{1.47}$$

eigenvector of the Hamiltonian with energy ω_k represents a possible quantum excitation of the lattice and it is called phonon. The generic state is given by

$$\frac{(a_{k1}^\dagger)^{N_{k1}}}{\sqrt{N_{k1}!}}\frac{(a_{k2}^\dagger)^{N_{k2}}}{\sqrt{N_{k2}!}}\ldots|0> \tag{1.48}$$

In conclusion the energy of an atom lattice is quantized. The phonon does not carry physical momentum however interacts with particles such as photons or neutrons as if it had a momentum k.

1.3 Continuum limit

The continuum limit of the oscillator lattice is obtained by taking the limit $N \to \infty$ or $a \to 0$ considering the length $L = aN$ finite.

In this limit $x_j = ja \to x$, $k = \chi/a$ is the wave vector.

$$u_i(t) = x_i' - x_i \to u(t,x) \tag{1.49}$$

$$\frac{u_{i+1}-u_i}{a} \to u'(t,x) \tag{1.50}$$

where

$$u'(t,x) = \frac{\partial u(t,x)}{\partial x} \tag{1.51}$$

The Lagrangian (1.7), in the continuum limit, becomes

$$\begin{aligned} L &= \frac{m}{2}\sum_{i=1}^{N}\dot{u}_i^2 - \frac{1}{2}C\sum_{i=1}^{N-1}(u_i - u_{i+1})^2 \\ &= \frac{m}{2a}\sum_{i=1}^{N}a\dot{u}_i^2 - \frac{1}{2}Ca^2\sum_{i=1}^{N-1}\frac{(u_i - u_{i+1})^2}{a^2} \\ &\to \frac{\mu}{2}\int_0^L dx\,\dot{u}(t,x)^2 - \frac{1}{2}K\int_0^L dx\,u'(t,x)^2 \end{aligned} \tag{1.52}$$

or

$$L = \int_0^L dx\,\mathcal{L}, \;\; \mathcal{L} = \frac{\mu}{2}\dot{u}(t,x)^2 - \frac{1}{2}Ku'(t,x)^2 \tag{1.53}$$

where we have introduced the linear density $\mu = m/a$ and the comprimibility modulus $K = Ca$. $\mathcal{L}$ is called Lagrangian density. Then the equations of motion (1.9) become the D'Alembert equation in one dimension

$$\mu\frac{d^2}{dt^2}u = K\lim_{a\to 0}\frac{(u_{i+1} - u_i) - (u_i - u_{i-1})}{a^2} = Ku'' \tag{1.54}$$

where $u'' = \partial^2 u/\partial x^2$. The velocity is given by

$$v = \sqrt{\frac{K}{\mu}} \tag{1.55}$$

In fact in the continuum limit we have

$$\omega_k^2 = 4\frac{K}{\mu a^2}\sin^2(\frac{ka}{2}) \to v^2k^2 \tag{1.56}$$

The expansion becomes

$$\begin{aligned} u(t,x) &= \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k a}}a_k u_j^{(k)}\exp\left(-i\omega_k t\right) + c.c. \\ &= \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L}}a_k \exp\left(-i(\omega_k t - kx_j)\right) + c.c. \\ &= \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L}}a_k \exp\left(-i(\omega_k t - kx)\right) + c.c. \end{aligned} \tag{1.57}$$

where the sum is extended over

$$k = \frac{2\pi n}{Na} = \frac{2\pi n}{L} \tag{1.58}$$

with $n = 0, \pm 1, \pm 2 \cdots$.

The total energy associated to the field is

$$H = \int dx\, \mathcal{H} \tag{1.59}$$

where the Hamiltonian density is given by

$$\mathcal{H} = \Pi \dot{u} - \mathcal{L} = \frac{1}{2\mu}\Pi^2 + \frac{1}{2}K(u')^2 \tag{1.60}$$

where

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{u}} \tag{1.61}$$

The quantization of the system is obtained by requiring the commutation relations at equal times

$$[u(t,x), \Pi(t,y)] = i\hbar\delta(x-y) \tag{1.62}$$

$$[u(t,x), u(t,y)] = [\Pi(t,x), \Pi(t,y)] = 0 \tag{1.63}$$

which imply for the operators $a_k, a_k^\dagger$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \tag{1.64}$$

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \tag{1.65}$$

The total Hamiltonian, using the expansion (1.57), becomes

$$H = \sum_k \hbar\omega_k (a_k^\dagger a_k + \frac{1}{2}) \tag{1.66}$$

Since the operator $a_k^\dagger a_k$ is definite positive, a state with minimum energy $|0>$ exists, defined by

$$a_k|0>= 0, \quad \forall k \tag{1.67}$$

This state corresponds to the fundamental state or of minimum energy. The generic state of the Hilbert space is built as a linear combination of the states (Fock states)

$$\frac{(a^\dagger_{k1})^{n_{k1}}}{\sqrt{n_{k1}!}}\frac{(a^\dagger_{k2})^{n_{k2}}}{\sqrt{n_{k2}!}}\dots(|0>) \tag{1.68}$$

The total energy of this state is the sum of the energy ω_{ki} of the single quantum excitations

$$E=\sum_i n_{ki}\omega_{ki} \tag{1.69}$$

2 Lagrangian field theory

2.1 Action and Euler equations for a continuous system

In general the action for a continuous system is given by:

$$S=\int_{t_i}^{t_f}dt\int_V d^3x\mathcal{L} \tag{2.1}$$

where $\mathcal{L}$ is the Lagrangian density with

$$\mathcal{L}=\mathcal{L}\left[\phi_A(t,\mathbf{x}),\dot{\phi}_A(t,\mathbf{x}),\partial_i\phi_A(t,\mathbf{x})\right],\quad i,j=1,2,3 \tag{2.2}$$

$\phi_A(t,\mathbf{x}),\ A=1\dots q$ are q fields and $\partial_i f\equiv\partial f/\partial x^i$. For Lagrangians depending on higher derivatives see [1] (Ostrogadski method).

The Action Principle requires the stationarity of the action for any variation $\delta\phi_A$ such that

$$\delta\phi_A|_{\partial V}=0,\ \ \delta\phi_A|_{t_i}=\delta\phi_A|_{t_f}=0 \tag{2.3}$$

Therefore (from now on the sum over equal indices is implied) we obtain

$$\begin{aligned}0=\delta S &= \int dtd^3x(\frac{\partial\mathcal{L}}{\partial\phi_A}\delta\phi_A+\frac{\partial\mathcal{L}}{\partial\dot{\phi}_A}\delta\dot{\phi}_A+\frac{\partial\mathcal{L}}{\partial\partial_j\phi_A}\delta\partial_j\phi_A)\\ &= \int dtd^3x(\frac{\partial\mathcal{L}}{\partial\phi_A}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_A}-\partial_j\frac{\partial\mathcal{L}}{\partial\partial_j\phi_A})\delta\phi_A\end{aligned}$$

$$+\int dtd^3x(\frac{\partial}{\partial t}(\frac{\partial\mathcal{L}}{\partial\dot{\phi}_A}\delta\phi_A)+\partial_j(\frac{\partial\mathcal{L}}{\partial\partial_j\phi_A}\delta\phi_A))$$
$$= \int dtd^3x(\frac{\partial\mathcal{L}}{\partial\phi_A}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_A}-\partial_j\frac{\partial\mathcal{L}}{\partial\partial_j\phi_A})\delta\phi_A \tag{2.4}$$

which implies the Euler equations

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\phi}_A}+\partial_j\frac{\partial\mathcal{L}}{\partial\partial_j\phi_A}=\frac{\partial\mathcal{L}}{\partial\phi_A} \tag{2.5}$$

It is easy, using (2.5), to get eqs.(1.54) from the Lagrangian given in eq. (1.53)

In the relativistic case the Lagrangian

$$\mathcal{L}=\mathcal{L}(\phi_A(x),\partial_\mu\phi_A(x)) \tag{2.6}$$

where $\mu=0,1,2,3$ and the Euler equations (2.5) become

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial_\mu\phi_A}=\frac{\partial\mathcal{L}}{\partial\phi_A} \tag{2.7}$$

The sum over μ is again understood.

Example. Schrödinger field. The Lagrangian for the Schrödinger field is given by (assuming $\hbar=1$):

$$\mathcal{L}=\frac{i}{2}(\dot{\psi}\psi^*-\psi\dot{\psi}^*)-\frac{1}{2m}\nabla\psi^*\nabla\psi-V\psi^*\psi \tag{2.8}$$

Using this Lagrangian in (2.5), one can derive the Schrödinger equations for ψ and ψ^*.

3 Quantization of the Klein-Gordon field

3.1 Quantization of the Klein-Gordon field in 1D

Let us consider the Klein-Gordon[1] real field in 1D, $x\in[0,L]$, obtained from the continuous string Lagrangian, eq.(1.53), assuming $\mu=1, K=1$ and

[1]The equation in 3D was proposed indipendently by O. Klein, W. Gordon, V. Fock and E. Schrödinger in 1926

adding a quadratic mass term. We assume the light velocity $c = 1$ and $\hbar = 1$.

$$\mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 - \phi'^2 - m^2\phi^2\right) \tag{3.9}$$

From the Euler equations we get

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2)\phi = 0 \tag{3.10}$$

The solutions are, using the expansion (1.57) with $\mu = 1$, $\omega_k \equiv E_k$

$$\phi(t, x) = \sum_k \sqrt{\frac{1}{2E_k L}}[a_k \exp(-i(E_k t - kx)) + h.c.] \tag{3.11}$$

with $k = 2\pi n/L, \;\; n = 0, \pm 1, \pm 2 \ldots$ and

$$E_k = \sqrt{k^2 + m^2} \tag{3.12}$$

The momentum density is given by

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \tag{3.13}$$

The Hamiltonian is defined

$$H = \int dx \mathcal{H} \tag{3.14}$$

with

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L} = \frac{1}{2}[\dot{\phi}^2 + \phi'^2 + m^2\phi^2] \tag{3.15}$$

In order to quantize the theory we promote ϕ and Π to self adjoint operators satisfying "equal time" commutation relations

$$[\phi(t, x), \Pi(t, y)] = i\delta(x - y) \tag{3.16}$$

$$[\phi(t, x), \phi(t, y)] = [\Pi(t, x), \Pi(t, y)] = 0 \tag{3.17}$$

We can now compute the commutation relations between the operators a_k and $a_k^\dagger$. Before let us show that

$$a_k = \frac{1}{\sqrt{2LE_k}} \int_0^L dx \exp(i(E_k t - kx))[E_k\phi + i\dot{\phi}] \tag{3.18}$$

In fact we have:

$$
\begin{aligned}
&\frac{1}{\sqrt{2LE_k}}\int_0^L dx\exp\left(i(E_kt-kx)\right)\Big[E_k\sum_{k'}\sqrt{\frac{1}{2E_{k'}L}}[a_{k'}\exp\left(-i(E_{k'}t-k'x)\right)\\
&+a^\dagger_{k'}\exp\left(i(E_{k'}t-k'x)\right)]\\
&+\sum_{k'}\sqrt{\frac{1}{2E_{k'}L}}E_{k'}[a_{k'}\exp\left(-i(E_{k'}t-k'x)\right)\\
&-a^\dagger_{k'}\exp\left(i(E_{k'}t-k'x)\right)]\Big]\\
&=\frac{1}{2L\sqrt{E_k}}\int_0^L dx\sum_{k'}\frac{1}{\sqrt{E_{k'}}}[(E_k+E_{k'})a_{k'}\exp\left(i((E_k-E_{k'})t-(k-k')x)\right)\\
&+(E_k-E_{k'})a^\dagger_{k'}\exp\left(i((E_k+E_{k'})t-(k+k')x)\right)\\
&=\frac{1}{2}\sum_{k'}\frac{1}{\sqrt{E_{k'}E_k}}[(E_k+E_{k'})a_{k'}\delta_{k,k'}\exp\left(i(E_k-E_{k'})t\right)\\
&+(E_k-E_{k'})a^\dagger_{k'}\delta_{k,-k'}\exp\left(i(E_k+E_{k'})t\right)]\\
&=a_k
\end{aligned}
\tag{3.19}
$$

where use has been made of

$$\frac{1}{L}\int_0^L dx\exp\left[i(k-q)x\right]=\delta_{kq} \tag{3.20}$$

Using (3.18) and (3.16)-(3.17), we have

$$
\begin{aligned}
&[a_k,a^\dagger_{k'}]=\delta_{kk'}\\
&[a_k,a_{k'}]=[a^\dagger_k,a^\dagger_{k'}]=0
\end{aligned}
\tag{3.21}
$$

Using the expansion for ϕ, given in eq.(3.11), and the commutation relations, we get

$$H=\sum_k E_k(a^\dagger_k a_k+\frac{1}{2}) \tag{3.22}$$

The generic state of the Hilbert space is built as a linear combination of the states (Fock states)

$$\frac{(a^\dagger_{k1})^{n_{k1}}}{\sqrt{n_{\mathbf{k1}}!}}\frac{(a^\dagger_{k2})^{n_{\mathbf{k2}}}}{\sqrt{n_{k2}!}}\dots(|0>) \tag{3.23}$$

and contains n_{k1} particles with energy E_{k1}, n_{k2} particles with energy E_{k2}... The fundamental state $|0>$ is such that

$$a_k|0>=0, \ \ \forall k \tag{3.24}$$

The state $|0>$ is called also the vacuum state because no particles are present. Note that the vacuum expectation value of the Hamiltonian is infinite:

$$<0|H|0>=\frac{1}{2}\sum_k E_k \tag{3.25}$$

However this is not a problem since usually one measures differences of energies between states.

3.2 Quantization of Klein-Gordon field in 3D

The action for the Klein-Gordon field in three spatial dimensions is obtained by generalizing the previous section

$$S=\int_{t_i}^{t_f} dt \int_V d^3x \frac{1}{2}\left(\dot{\phi}^2-(\nabla\phi)^2-m^2\phi^2\right) \tag{3.26}$$

where $V\equiv L^3$. This action can be written also in the covariant form

$$S=\int_{V_4} d^4x \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi-m^2\phi^2\right) \tag{3.27}$$

where V_4 is now a space-time volume, d^4x is the space time volume element and $\mu=0,1,2,3$. The sum over the repeated indices μ is understood. Assuming the mass as fundamental dimension, the dimension of the Klein-Gordon field ar M^1 so that the action is dimensionless. From the Euler equations we get the differential equations

$$(\frac{\partial^2}{\partial t^2}-\nabla^2+m^2)\phi=0 \tag{3.28}$$

or in covariant form

$$(\partial^\mu\partial_\mu+m^2)\phi\equiv(\Box+m^2)\phi=0 \tag{3.29}$$

This equation is the simplest relativistic estension of the Schrödinger equation, obtained by replacing the relativistic mass condition

$$(E_k^2 - \mathbf{k}^2 - m^2) = 0 \tag{3.30}$$

with the differential equation (3.28) obtained by substituting the classical variables E_k and $\mathbf{k}$ by the operators

$$E_k \to i\frac{\partial}{\partial t}, \quad \mathbf{k} \to -i\nabla \tag{3.31}$$

As we will see the quantization of this field theory describe a many boson relativistic theory.

To quantize this theory, we postulate equal time commutation relations between the operators ϕ and Π are

$$[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = i\delta^3(x-y) \tag{3.32}$$

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = [\Pi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = 0 \tag{3.33}$$

The field expansion is now

$$\phi(t,\mathbf{x}) = \sum_{\mathbf{k}} \sqrt{\frac{1}{2E_k L^3}}[a_{\mathbf{k}} \exp\left(-i(E_k t - \mathbf{k}\cdot\mathbf{x})\right) + h.c.] \tag{3.34}$$

with the dispersion relation of a relativistic particle

$$E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} \tag{3.35}$$

and

$$k^i = \frac{2\pi n_i}{L}, \quad i = 1,2,3, n_i = 0, \pm 1, \pm 2, \dots \tag{3.36}$$

The inversion relation for the operators $a_{\mathbf{k}}$ is now

$$a_{\mathbf{k}} = \frac{1}{\sqrt{2L^3 E_k}} \int_V d^3x \exp\left(i(E_k t - \mathbf{k}\cdot\mathbf{x})\right)[E_k\phi + i\dot{\phi}] \tag{3.37}$$

and the commutations relations are trivial generalization of eqs.(3.21).

The Hamiltonian is

$$H = \int d^3x \mathcal{H} = \int d^3x \frac{1}{2}[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2] \tag{3.38}$$

and, using the field expansion (3.34), it turns out to be

$$H = \sum_{\mathbf{k}} E_k (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{2}) \tag{3.39}$$

The Hilbert space is built as in the previous section, with suitable generalizations. For instance the state with $n_{\mathbf{k}}$ particles with energy E_k is given by

$$\frac{(a^\dagger_{\mathbf{k}})^{n_{\mathbf{k}}}}{\sqrt{n_{\mathbf{k}}!}} |0> \tag{3.40}$$

As we will see in the following these particles have momentum $\mathbf{k}$ and spin 0 (bosons).

Let us finally note that when working in $\mathbb{R}^3$, the expansion becomes a Fourier transform

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \frac{1}{\sqrt{2E_k}} \left[a(\mathbf{k}) e^{-ikx} + h.c. \right] \tag{3.41}$$

where now x and k denote, using the covariant notations, the fourvectors and

$$kx = k^0 x^0 - \mathbf{k} \cdot \mathbf{x} \equiv E_k t - \mathbf{k} \cdot \mathbf{x} \tag{3.42}$$

3.3 Noether Theorem

Let us now prove the Noether theorem (1918)[2]. The theorem states that to every continuous transformation for which the action is invariant ($\delta S = 0$), there corresponds a definite function which is time conserved. This function or the corresponding operator, after quantization, is the generator of the corresponding infinitesimal transformation.

Let us consider a generic action (we use natural units, $\hbar = c = 1$)

$$S_V = \int_V d^4x \mathcal{L}(\phi_A, \partial_\mu \phi_A) \, , \, A = 1, \cdots q, \tag{3.43}$$

[2]Emmy Noether, 1882-1935

where V is a space-time volume and $\phi_A(x),\ A = 1, \cdots q$, denote a field with q components.We are going to consider both an infinitesimal variation of form of the fields and of the space-time coordinates

$$\phi_A \to \tilde{\phi}_A = \phi_A + \delta\phi_A, \ \ x \to x' = x + \delta x \tag{3.44}$$

In general the total variation of a function is defined as

$$\Delta f = f'(x') - f(x) = f'(x') - f(x') + f(x') - f(x) \tag{3.45}$$

To first order we get

$$\Delta f \sim \delta f + \frac{\partial f}{\partial x}\delta x \tag{3.46}$$

where we have defined the local variation

$$\delta f = f'(x) - f(x) \tag{3.47}$$

Let us suppose that the action is invariant under (3.44)

$$S'_{V'} = \int d^4x' \mathcal{L}'(x') = S_V \tag{3.48}$$

where

$$\mathcal{L}'(x') = \mathcal{L}'(\tilde{\phi}_A(x'), \partial'_\mu \tilde{\phi}_A(x')) = \mathcal{L}(x) + \Delta\mathcal{L}(x) \tag{3.49}$$

with

$$\begin{aligned} \Delta\mathcal{L}(x) &= \mathcal{L}'(x') - \mathcal{L}(x') + \mathcal{L}(x') - \mathcal{L}(x) \\ &\sim \frac{\partial\mathcal{L}}{\partial\phi_A}\delta\phi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\delta\partial_\mu\phi_A + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu \end{aligned} \tag{3.50}$$

with $\delta\phi_A$ and $\delta\partial_\mu\phi_A$ the local variations

$$\delta\phi_A = \phi'_A(x) - \phi_A(x) \tag{3.51}$$

The variation of the action can be written as

$$\delta S_V = \int_{V'} d^4x' \mathcal{L}'(x') - \int_V d^4x \mathcal{L}(x) \tag{3.52}$$

or

$$\begin{aligned}
\delta S_V &= \int_{V'} d^4x'\mathcal{L}(x) + \int_{V'} d^4x'\Delta\mathcal{L}(x) - \int_V d^4x\mathcal{L}(x) \\
&\sim \int_V d^4x\mathcal{L}(x)|\frac{\partial x'}{\partial x}| + \int_V d^4x\Delta\mathcal{L}(x) - \int_V d^4x\mathcal{L}(x) \\
&\sim \int_V d^4x\mathcal{L}(x)(1+\partial_\mu\delta x^\mu) - \int_V d^4x\mathcal{L}(x) \\
&\quad + \int_V d^4x[\frac{\partial\mathcal{L}}{\partial\phi_A}\delta\phi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\delta\partial_\mu\phi_A + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu] \\
&\sim \int_V d^4x[\frac{\partial\mathcal{L}}{\partial\phi_A} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A})]\delta\phi_A \\
&\quad + \int_V d^4x\partial_\mu[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\delta\phi_A + \mathcal{L}\delta x^\mu] \qquad (3.53)
\end{aligned}$$

In the previous equations the Jacobian has been computed to first order in Δ as

$$\det(I+\Delta) = \exp[\ln\det(1+\Delta)] = \exp[\mathrm{Tr}\ln(1+\Delta)] \sim \exp[\mathrm{Tr}\Delta] \sim 1 + \mathrm{Tr}\Delta \qquad (3.54)$$

with

$$\Delta^\mu_\nu = \partial_\nu\delta x^\mu \qquad (3.55)$$

Using the Euler equations of motion in (3.53), we obtain

$$\int_V d^4x\partial_\mu[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\delta\phi_A + \mathcal{L}\delta x^\mu] = 0 \qquad (3.56)$$

By considering the global variation

$$\Delta\phi_A = \tilde{\phi}_A(x') - \phi_A(x) \qquad (3.57)$$

we have

$$\Delta\phi_A \sim \delta\phi_A + \partial_\mu\phi_A\delta x^\mu \qquad (3.58)$$

In term of the global variation $\Delta\phi_A$ we get

$$\int_V d^4x\partial_\mu[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\Delta\phi_A + \mathcal{L}\delta x^\mu - \delta x^\nu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\partial_\nu\phi_A] = 0 \qquad (3.59)$$

and since the variations are arbitrary

$$\partial_\mu [\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A} \Delta \phi_A + \mathcal{L} \delta x^\mu - \delta x^\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A} \partial_\nu \phi_A] = 0 \tag{3.60}$$

Let us now study the consequences of the Noether theorem in some examples.

Space-time Translations Let us consider an infinitesimal space time translation

$$\delta x^\mu = \epsilon^\mu \tag{3.61}$$

Under this transformation the fields are invariant

$$\Delta \phi_A = 0 \tag{3.62}$$

From eq.(3.60) we get

$$\partial_\mu [\mathcal{L} \epsilon^\mu - \epsilon^\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A} \partial_\nu \phi_A] = 0 \tag{3.63}$$

Being ϵ^μ arbitrary we get

$$\partial_\mu T^{\mu\nu} = 0 \tag{3.64}$$

where

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_A} \partial^\nu \phi_A - g^{\mu\nu} \mathcal{L} \tag{3.65}$$

By integrating over d^3x the $\mu = 0$ component of $T^{\mu\nu}$, we get four invariants corresponding to the total four-momentum of the field

$$P^\nu = \int d^3x T^{0\nu} \tag{3.66}$$

In particular for $\nu = 0$ we recover the Hamiltonian. For $\nu = i$, we get the spatial momentum associated to the field

$$P^i = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot{\phi}_A} \partial^i \phi_A \tag{3.67}$$

For example, in the case of the Klein-Gordon field,

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \tag{3.68}$$

and, using the field expansion (3.34), we obtain

$$P^i = \int d^3x \dot{\phi} \partial^i \phi = \sum_{\mathbf{k}} k^i a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \tag{3.69}$$

In conclusion each quantum of the Klein-Gordon field contributes to the total momentum of the field with its momentum **k**.

Lorentz Transformations Let us then consider an infinitesimal Lorentz transformation

$$\Lambda^{\mu}_{\cdot\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}_{\cdot\nu} \tag{3.70}$$

The matrix condition

$$\Lambda^T g \Lambda = g \tag{3.71}$$

implies that the tensor $\epsilon^{\mu\nu}$ is antisymmetric,

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \tag{3.72}$$

Let as assume that under the Lorentz transformation the fields transform as

$$\Delta \phi_A = -\frac{1}{2} \Sigma^{\mu\nu}_{AB} \epsilon_{\mu\nu} \phi_B \tag{3.73}$$

Let us consider some examples. For the Klein-Gordon field we have

$$\phi'(x') = \phi(x(x')) \tag{3.74}$$

or the field is a scalar field. For vector fields, like the four potential $A^\mu(x)$,we have

$$A'^{\mu}(x') = \Lambda^{\mu}_{\cdot\nu} A^{\nu}(x(x')) \tag{3.75}$$

By substituting (3.73) in (3.60), we obtain

$$\begin{aligned} 0 &= \partial_\mu[-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\frac{1}{2}\Sigma^{\rho\sigma}_{AB}\epsilon_{\rho\sigma}\phi_B + \mathcal{L}\epsilon^{\mu\sigma}x_\sigma - \epsilon^{\nu\sigma}x_\sigma\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\partial_\nu\phi_A] \\ &= \partial_\mu[-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\frac{1}{2}\Sigma^{\rho\sigma}_{AB}\epsilon_{\rho\sigma}\phi_B + \mathcal{L}\epsilon_{\rho\sigma}g^{\mu\rho}x^\sigma - \epsilon_{\rho\sigma}x^\sigma\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\partial^\rho\phi_A] \\ &= \partial_\mu[-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\frac{1}{2}\Sigma^{\rho\sigma}_{AB}\epsilon_{\rho\sigma}\phi_B - \epsilon_{\rho\sigma}x^\sigma T^{\rho\mu}] \\ &= \partial_\mu[-\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\frac{1}{2}\Sigma^{\rho\sigma}_{AB}\epsilon_{\rho\sigma}\phi_B - \frac{1}{2}\epsilon_{\rho\sigma}(x^\sigma T^{\rho\mu} - x^\rho T^{\sigma\mu})] \end{aligned} \tag{3.76}$$

or

$$\partial_\mu\mathcal{M}^{\mu\rho\sigma} = 0 \tag{3.77}$$

with

$$\mathcal{M}^{\mu\rho\sigma} = x^\rho T^{\sigma\mu} - x^\sigma T^{\rho\mu} - \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\Sigma^{\rho\sigma}_{AB}\phi_B \tag{3.78}$$

In conclusion we can built the six invariants

$$M^{\rho\sigma} = \int d^3x\mathcal{M}^{0\rho\sigma} \tag{3.79}$$

M^{ij} are the angular momentum components or the generators of the $O(3)$ rotations, while M^{0i} are the generators of the Lorentz transformations.

Internal transformations In this case only the field transforms, $\delta x = 0$, $\Delta\phi = \delta\phi$. Therefore from eq. (3.60), we get

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_A}\Delta\phi_A = 0 \tag{3.80}$$

As an example let us consider the gauge transformations for the Schrödinger Lagrangian

$$\mathcal{L} = \frac{i}{2}(\phi^*\dot{\phi} - \dot{\phi}^*\phi) - \frac{1}{2m}\nabla\phi^*\nabla\phi - V\phi^*\phi \tag{3.81}$$

The Lagrangian is invariant under the internal transformation

$$\phi \to \exp(i\alpha)\phi \tag{3.82}$$

where α is a real number. Therefore for infinitesimal transformation

$$\Delta\phi \sim i\alpha\phi \tag{3.83}$$

The Noether theorem implies

$$\left[(\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \partial_k \frac{\partial \mathcal{L}}{\partial \partial_k \phi})\Delta\phi + (\phi \to \phi^*) \right] = 0 \tag{3.84}$$

But

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{i}{2}\phi^*, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = -\frac{i}{2}\phi \tag{3.85}$$

$$\frac{\partial \mathcal{L}}{\partial \partial_k \phi} = -\frac{1}{2m}\partial_k \phi^*, \quad \frac{\partial \mathcal{L}}{\partial \partial_k \phi^*} = -\frac{1}{2m}\partial_k \phi \tag{3.86}$$

and substituting in (3.84), we get

$$\alpha \left[\frac{\partial}{\partial t}(-\phi^*\phi) + \partial_k(-\frac{i}{2m}\partial_k\phi^*\phi + \frac{i}{2m}\phi\partial_k\phi^*) \right] = 0 \tag{3.87}$$

or

$$\frac{\partial}{\partial t}(-\phi^*\phi) + \partial_k(-\frac{i}{2m}\partial_k\phi^*\phi + \frac{i}{2m}\phi\partial_k\phi^*) = 0 \tag{3.88}$$

In conclusion, we obtain the continuity equation

$$\frac{\partial}{\partial t}(\phi^*\phi) + \nabla(+\frac{i}{2m}(\nabla\phi^*)\phi - \frac{i}{2m}\phi\nabla\phi^*) = 0 \tag{3.89}$$

4 Quantization of the electromagnetic field

4.1 Lagrangian and Hamiltonian

Let us start by writing the Lagrangian of the electromagnetic field (we work in the Heaviside-Lorentz system[3]):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}F_{0i}F^{0i} - \frac{1}{4}F_{ij}F^{ij} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{4.1}$$

[3]In this electromagnetic system $\alpha = e^2/4\pi\hbar c = e^2/4\pi$ in natural units

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.2}$$

In particular, in eq.(4.1) we have used $F^{0i} = -E^i$ and $F^{ij} = -\epsilon_{ijk} B^k$. From this Lagrangian we can derive the Euler equations

$$\frac{\partial \mathcal{L}}{\partial A_\rho} = \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial \partial_\sigma A_\rho} \right) \tag{4.3}$$

Now

$$\frac{\partial \mathcal{L}}{\partial \partial_\sigma A_\rho} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \frac{\partial F_{\mu\nu}}{\partial \partial_\sigma A_\rho} = -F^{\sigma\rho} \tag{4.4}$$

and therefore we get the free equations of motion

$$\partial_\sigma F^{\sigma\rho} = 0 \tag{4.5}$$

In presence of an interaction with a corrent $\jmath^\mu$, the Lagrangian becomes

$$\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_I \tag{4.6}$$

$$\mathcal{L}_I = -j_\mu A^\mu \tag{4.7}$$

and the Euler equations of motion become the Maxwell equations in presence of the current j^μ

$$\partial_\sigma F^{\sigma\rho} = j^\rho \tag{4.8}$$

To quantize the system, let us first compute the Hamiltonian. Let define the momentum density as

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \tag{4.9}$$

Using the eq. (4.4), we get

$$\Pi_\mu = -F_{0\mu} \tag{4.10}$$

Therefore the zero component $\Pi_0 = 0$ and

$$\Pi_i = -F_{0i} = -E^i = \dot{A}^i + \partial_i A^0 \tag{4.11}$$

The Hamiltonian density

$$\mathcal{H} = \Pi_i \dot{A}^i - \mathcal{L} \tag{4.12}$$

and the Hamiltonian

$$H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x (\mathbf{\Pi}^2 + \mathbf{B}^2) - \int d^3x \Pi_i \partial_i A^0 \tag{4.13}$$

The last term can be integrated by parts, neglecting the boundary terms,

$$-\int d^3x \Pi_i \partial_i A^0 = \int d^3x \partial_i \Pi_i A^0 = 0 \tag{4.14}$$

where use has been done of

$$\nabla \cdot \mathbf{E} = 0 \tag{4.15}$$

The Lagrangian (4.1) is invariant under gauge transformations

$$A^\mu \to A^\mu + \partial^\mu \chi \tag{4.16}$$

where χ is an arbitrary scalar function. In order to show that the Lagrangian is invariant, it is sufficient to observe that

$$F^{\mu\nu} \to F^{\mu\nu} + \partial^\mu \partial^\nu \chi - \partial^\nu \partial^\mu \chi = F^{\mu\nu} \tag{4.17}$$

We can use this arbitrariness to perform the quantization of the electromagnetic field in different gauges, choosing a condition on the field A^μ. Here we choose the Coulomb gauge[4]

$$\nabla \cdot \mathbf{A} = 0 \tag{4.18}$$

In this gauge

$$\nabla \cdot \mathbf{E} = 0 = \partial_i \dot{A}^i + \triangle A^0 = \triangle A^0 \tag{4.19}$$

In conclusion, eq.(4.15) implies $A^0 = 0$. The gauge

$$\nabla \cdot \mathbf{A} = \mathbf{0}, \quad A^0 = 0 \tag{4.20}$$

is called *radiation gauge*. Working in this gauge we loose manifest covariance. The advantage is that we work with the two independent degrees of freedom, as we will see when expanding the field in normal modes. A^0 and Π^0 will not be quantized and the field $\mathbf{A}$ is transverse, due to eq.(4.20).

We can now expand the field in normal modes in a finite volume V

$$\mathbf{A}(x) = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{1}{\sqrt{2V\omega_k}} \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} [a^\alpha_{\mathbf{k}} e^{-ikx} + h.c.], \quad k_i = n_i \frac{2\pi}{L} \tag{4.21}$$

[4]The quantization can be performed also in the Lorenz gauge. See for example [15].

where

$$kx \equiv k^0 x^0 - \mathbf{k} \cdot \mathbf{x} \tag{4.22}$$

and $k^0 \equiv \omega_k = |\mathbf{k}|$. The polarization vectors $\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}$ are orthonormal and transverse to $\mathbf{k}$. For simplicity we assume real polarization vectors.

$$\boldsymbol{\epsilon}^\alpha_{\mathbf{k}} \cdot \boldsymbol{\epsilon}^\beta_{\mathbf{k}} = \delta^{\alpha\beta}, \quad \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} \cdot \mathbf{k} = 0 \tag{4.23}$$

Furthermore $\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}$ and $\mathbf{k}/|\mathbf{k}|$ are a complete basis or

$$\sum_{\alpha=1,2} \epsilon^{\alpha i} \epsilon^{\alpha j} + \frac{k^i k^j}{\mathbf{k}^2} = \delta^{ij} \tag{4.24}$$

The Hamiltonian is

$$H = \frac{1}{2} \int d^3x [\mathbf{\Pi}^2 + (\mathbf{\nabla} \times \mathbf{A})^2] \tag{4.25}$$

where

$$\mathbf{\Pi} = \dot{\mathbf{A}} \tag{4.26}$$

The quantization[5] is obtained by requiring the commutation relations

$$[a^\alpha_{\mathbf{k}}, a^{\beta\dagger}_{\mathbf{k}'}] = \delta^{\alpha\beta} \delta_{\mathbf{k},\mathbf{k}'} \tag{4.27}$$

$$[a^\alpha_{\mathbf{k}}, a^\beta_{\mathbf{k}'}] = [a^{\alpha\dagger}_{\mathbf{k}}, a^{\beta\dagger}_{\mathbf{k}'}] = 0 \tag{4.28}$$

By substituting in the Hamiltonian the expansion in normal modes (4.21) we get

$$H = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \omega_k (a^{\alpha\dagger}_{\mathbf{k}} a^\alpha_{\mathbf{k}} + \frac{1}{2}) \tag{4.29}$$

Proof:

[5] The first attempt of quantization of the electromagnetic field was performed by M. Born, W. Heisenberg and P. Jordan in 1926. Then in 1927 Dirac published the paper on *The quantum theory of the emission and absorption of radiation.* The idea of the second quantization was also proposed by Jordan, in 1927, while the expression was coined by Dirac. The general theory of quantum fields through the method of canonical quantization was presented in W. Heisenberg and W. Pauli in 1929.

Let us first consider

$$\dot{\mathbf{A}}(x) = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{1}{\sqrt{2V\omega_k}} (-i\omega_k) \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} [a^\alpha_{\mathbf{k}} e^{-ikx} - h.c.] \tag{4.30}$$

and

$$\boldsymbol{\nabla} \times \mathbf{A}(x) = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{1}{\sqrt{2V\omega_k}} (i\mathbf{k}) \times \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} [a^\alpha_{\mathbf{k}} e^{-ikx} - h.c.] \tag{4.31}$$

and compute

$$\begin{aligned}
\int d^3x (\dot{\mathbf{A}})^2 &= \int d^3x \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{1}{\sqrt{2V\omega_k}} (-i\omega_k) \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} [a^\alpha_{\mathbf{k}} e^{-ikx} - h.c.] \\
& \cdot \sum_{\mathbf{k}'} \sum_{\alpha'=1,2} \frac{1}{\sqrt{2V\omega_{k'}}} (-i\omega_{k'}) \boldsymbol{\epsilon}^{\alpha'}_{\mathbf{k}'} [a^{\alpha'}_{\mathbf{k}'} e^{-ik'x} - h.c.] \\
&= -\sum_{\mathbf{k}} \sum_{\alpha=1,2} \sum_{\mathbf{k}'} \sum_{\alpha'=1,2} \frac{1}{2} \sqrt{\omega_k \omega_{k'}} \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} \cdot \boldsymbol{\epsilon}^{\alpha'}_{\mathbf{k}'} \\
& \left\{ \left[a^\alpha_{\mathbf{k}} a^{\alpha'}_{\mathbf{k}'} \delta_{\mathbf{k},-\mathbf{k}'} \exp\left[-i(\omega_k + \omega_{k'})t\right] + h.c. \right] - \right. \\
& \left. \left[a^\alpha_{\mathbf{k}} a^{\alpha'\dagger}_{\mathbf{k}'} \delta_{\mathbf{k},\mathbf{k}'} \exp\left[-i(\omega_k - \omega_{k'})t\right] + h.c. \right] \right\} \\
&= -\frac{1}{2} \sum_{\mathbf{k},\alpha,\alpha'} \omega_k \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} \cdot \boldsymbol{\epsilon}^{\alpha'}_{-\mathbf{k}} \left[a^\alpha_{\mathbf{k}} a^{\alpha'}_{-\mathbf{k}} \exp\left(-2i\omega_k t\right) + h.c. \right] \\
&+ \frac{1}{2} \sum_{\mathbf{k},\alpha,\alpha'} \omega_k \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} \cdot \boldsymbol{\epsilon}^{\alpha'}_{\mathbf{k}} \left[a^\alpha_{\mathbf{k}} a^{\alpha'\dagger}_{\mathbf{k}} + h.c. \right] \\
&= -\frac{1}{2} \sum_{\mathbf{k},\alpha,\alpha'} \omega_k \boldsymbol{\epsilon}^\alpha_{\mathbf{k}} \cdot \boldsymbol{\epsilon}^{\alpha'}_{-\mathbf{k}} \left[a^\alpha_{\mathbf{k}} a^{\alpha'}_{-\mathbf{k}} \exp\left(-2i\omega_k t\right) + h.c. \right] \\
&+ \frac{1}{2} \sum_{\mathbf{k},\alpha} \omega_k \left[a^\alpha_{\mathbf{k}} a^{\alpha\dagger}_{\mathbf{k}} + h.c. \right]
\end{aligned} \tag{4.32}$$

where we have used eq.(4.23). Furthermore we get

$$\int d^3x (\boldsymbol{\nabla} \times \mathbf{A})^2 = \int d^3x \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{1}{\sqrt{2V\omega_k}} (i\mathbf{k} \times \boldsymbol{\epsilon}^\alpha_{\mathbf{k}}) [a^\alpha_{\mathbf{k}} e^{-ikx} - h.c.]$$

$$
\begin{aligned}
& \cdot \sum_{\mathbf{k}'} \sum_{\alpha'=1,2} \frac{1}{\sqrt{2V\omega_{k'}}} (i\mathbf{k}' \times \boldsymbol{\epsilon}_{\mathbf{k}'}^{\alpha'})[a_{\mathbf{k}'}^{\alpha'} e^{-ik'x} - h.c.] \\
= \quad & -\sum_{\mathbf{k}} \sum_{\alpha=1,2} \sum_{\mathbf{k}'} \sum_{\alpha'=1,2} \frac{1}{2} \frac{1}{\sqrt{\omega_k \omega_{k'}}} (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}) \cdot (\mathbf{k}' \times \boldsymbol{\epsilon}_{\mathbf{k}'}^{\alpha'}) \\
& \Big\{ \Big[a_{\mathbf{k}}^{\alpha} a_{\mathbf{k}'}^{\alpha'} \delta_{\mathbf{k},-\mathbf{k}'} \exp\left[-i(\omega_k + \omega_{k'})t\right] + h.c. \Big] - \\
& \Big[a_{\mathbf{k}}^{\alpha} a_{\mathbf{k}'}^{\alpha'\dagger} \delta_{\mathbf{k},\mathbf{k}'} \exp\left[-i(\omega_k - \omega_{k'})t\right] + h.c. \Big] \Big\} \\
= \quad & -\frac{1}{2} \sum_{\mathbf{k},\alpha,\alpha'} \frac{1}{\omega_k} \Big\{ -(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}) \cdot (\mathbf{k} \times \boldsymbol{\epsilon}_{-\mathbf{k}}^{\alpha'}) \Big[a_{\mathbf{k}}^{\alpha} a_{\mathbf{k}'}^{\alpha'} \delta_{\mathbf{k},-\mathbf{k}'} \exp\left[-2i\omega_k t\right] + h.c. \Big] - \\
& (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}) \cdot (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha'}) \Big[a_{\mathbf{k}}^{\alpha} a_{\mathbf{k}'}^{\alpha'\dagger} \delta_{\mathbf{k},\mathbf{k}'} + h.c. \Big] \Big\} \\
= \quad & \frac{1}{2} \sum_{\mathbf{k},\alpha,\alpha'} \omega_k \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha} \cdot \boldsymbol{\epsilon}_{-\mathbf{k}}^{\alpha'} \Big[a_{\mathbf{k}}^{\alpha} a_{-\mathbf{k}}^{\alpha'} \exp\left(-2i\omega_k t\right) + h.c. \Big] \\
+ \quad & \frac{1}{2} \sum_{\mathbf{k},\alpha} \omega_k \Big[a_{\mathbf{k}}^{\alpha} a_{\mathbf{k}}^{\alpha\dagger} + h.c. \Big]
\end{aligned}
\tag{4.33}
$$

where we have used

$$
(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}) \cdot (\mathbf{k} \times \boldsymbol{\epsilon}_{-\mathbf{k}}^{\alpha'}) = \mathbf{k}^2 \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha} \cdot \boldsymbol{\epsilon}_{-\mathbf{k}}^{\alpha'} \tag{4.34}
$$

$$
(\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}) \cdot (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha'}) = \mathbf{k}^2 \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha} \cdot \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha'} = \delta_{\alpha\alpha'} \mathbf{k}^2 \tag{4.35}
$$

Summing eq. (4.32) and eq. (4.33), we obtain

$$
\begin{aligned}
H \quad &= \quad \frac{1}{2} \int d^3x [\dot{\mathbf{A}}^2 + (\boldsymbol{\nabla} \times \mathbf{A})^2] \\
&= \quad \frac{1}{2} \sum_{\mathbf{k},\alpha} \omega_k \Big[a_{\mathbf{k}}^{\alpha} a_{\mathbf{k}}^{\alpha\dagger} + h.c. \Big] \\
&= \quad \sum_{\mathbf{k},\alpha} \omega_k (a_{\mathbf{k}}^{\alpha\dagger} a_{\mathbf{k}}^{\alpha} + \frac{1}{2})
\end{aligned}
\tag{4.36}
$$

Using the Noether theorem we can build the energy momentum tensor

$$
T_i^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} \partial_i A^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^k} \partial_i A^k = \Pi_k \partial_i A^k = \dot{A}^k \partial_i A^k \tag{4.37}
$$

Therefore the momentum of the field is

$$P^i = \int d^3x T^{0i} = -\int d^3x \dot{A}^k \partial_i A^k \tag{4.38}$$

By substituting the expression in normal modes (4.21), we obtain

$$\mathbf{P} = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \mathbf{k}(a_{\mathbf{k}}^{\alpha\dagger} a_{\mathbf{k}}^{\alpha}) \tag{4.39}$$

This expression allows us to associate to the photon state $a_{\mathbf{k}}^{\alpha\dagger}|0>$ the momentum $\mathbf{k}$ and therefore the mass of the photon is zero

$$m^2 = \omega_k^2 - \mathbf{k}^2 = 0 \tag{4.40}$$

Note that the usual expression of the Poynting vector is equivalent to (4.37), since

$$\begin{aligned} \int d^3x(\mathbf{E}\times\mathbf{B})^i &= -\int d^3x \epsilon_{ijk}\dot{A}^j \epsilon_{klm}\partial_l A^m \\ &= -\int d^3x \dot{A}^j \partial_l A^m [\delta_{il}\delta_{jm} - (l \leftrightarrow m)] \\ &= -\int d^3x (\dot{A}^m \partial_i A^m - \dot{A}^l \partial_l A_i) \\ &= -\int d^3x \dot{A}^m \partial_i A^m \end{aligned} \tag{4.41}$$

where in the last step we have eliminated a term by integrating by parts and taking into account the Coulomb gauge condition.

The Hilbert space is built by many photons states

$$|n_{\mathbf{k}_1,\alpha_1}, n_{\mathbf{k}_2,\alpha_2} \ldots > \tag{4.42}$$

with

$$|n_{\mathbf{k}_1,\alpha_1}, n_{\mathbf{k}_2,\alpha_2} \ldots >= \frac{(a^\dagger_{\mathbf{k}_1,\alpha_1})^{n_{\mathbf{k}_1,\alpha_1}}}{\sqrt{n_{\mathbf{k}_1,\alpha_1}!}} \frac{(a^\dagger_{\mathbf{k}_2,\alpha_2})^{n_{\mathbf{k}_2,\alpha_2}}}{\sqrt{n_{\mathbf{k}_2,\alpha_2}!}} \ldots |0> \tag{4.43}$$

with

$$a_{\mathbf{k},\alpha}|0>= 0, \ \ \forall\alpha, \mathbf{k} \tag{4.44}$$

In conclusion, neglecting the vacuum energy, the total energy and the total momentum of the electromagnetic field are the sum of single photon contributions of energy ω_k and momentum $\mathbf{k}$.

$$H|n_{\mathbf{k}_1,\alpha_1}, n_{\mathbf{k}_2,\alpha_2} \ldots >= \sum_i \omega_{\mathbf{k}_i} n_{\mathbf{k}_i,\alpha_i} |n_{\mathbf{k}_1,\alpha_1}, n_{\mathbf{k}_2,\alpha_2} \ldots > \tag{4.45}$$

$$\mathbf{P}|n_{\mathbf{k}_1,\alpha_1}, n_{\mathbf{k}_2,\alpha_2} \ldots >= \sum_i \mathbf{k}_i n_{\mathbf{k}_i,\alpha_i} |n_{\mathbf{k}_1,\alpha_1}, n_{\mathbf{k}_2,\alpha_2} \ldots > \tag{4.46}$$

The photon is characterized by the momentum $\mathbf{k}$ and by the polarization α. From the two one photon states

$$a^\dagger_{\mathbf{k},1}|0>, a^\dagger_{\mathbf{k},2}|0> \tag{4.47}$$

we can define the circular polarization states

$$\mp \frac{1}{\sqrt{2}}[a^\dagger_{\mathbf{k},1} \mp i a^\dagger_{\mathbf{k},2}]|0> \tag{4.48}$$

These two states are eigenvectors of the helicity operator, the projection of the spin of the photon along the photon flight direction), with eigenvalues ± 1 (see for example [2, 17]) . In conclusion the photon has spin one but only states with helicity ± 1 are allowed. This is related to the fact that the photon is massless and it is a consequence of the theory of Poincaré representations.

Using the expansion (4.21) and the commutation relations (4.28) we can compute the commutation relations between the canonical operators field and momentum density:

$$\begin{aligned}
[A^i(t,\boldsymbol{x}), \Pi^j(t,\boldsymbol{y})] &= \sum_{\mathbf{k}} \sum_{\alpha=1,2} \sum_{\mathbf{k}'} \sum_{\alpha'=1,2} \frac{1}{\sqrt{2V\omega_k}} \frac{1}{\sqrt{2V\omega_{k'}}} (-i\omega_{k'}) \epsilon^{\alpha i}_{\mathbf{k}} \epsilon^{\alpha' j}_{\mathbf{k}'} \\
&\quad [a^\alpha_{\mathbf{k}} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + h.c., a^{\alpha'}_{\mathbf{k}'} e^{-i(\omega_{k'} t - \mathbf{k}'\cdot\mathbf{y})} - h.c.] \\
&= \frac{i}{2V} \sum_{\mathbf{k},\alpha} \epsilon^{\alpha i}_{\mathbf{k}} \epsilon^{\alpha j}_{\mathbf{k}} [e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + h.c.] \\
&= \frac{i}{2V} \sum_{\mathbf{k}} (\delta^{ij} - \frac{k^i k^j}{k^2}) [e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + h.c.] \\
&= \frac{i}{V} \sum_{\mathbf{k}} (\delta^{ij} - \frac{k^i k^j}{k^2}) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}
\end{aligned}$$

$$= i[\delta^{ij}\delta^3(x-y) - \frac{\partial^i\partial^j}{\nabla^2}\delta^3(x-y)]$$

$$= i[\delta^{ij}\delta^3(x-y) + \partial^i\partial^j\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}] \quad (4.49)$$

where we have used eq.(4.24) and introduced the distribution $[\nabla^2]^{-1}\delta^3(x-y)$. Since

$$\nabla^2 E(\mathbf{x}-\mathbf{y}) = \delta^3(x-y) \quad (4.50)$$

with

$$E(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \quad (4.51)$$

$$[\nabla^2]^{-1}\delta^3(x-y) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \quad (4.52)$$

Proceeding in similar way we can show that

$$[A^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})] = [\Pi^i(t,\boldsymbol{x}), \Pi^j(t,\boldsymbol{y})] = 0 \quad (4.53)$$

The three **A** operators are not independent, since we are quantizing in the Coulomb gauge. As a consequence the commutator (4.49) between the field and the momentum density is not canonical (see [3]).

4.2 Casimir effect

The Casimir effect is the macroscopic manifestation of the vacuum fluctuations of the quantized electromagnetic field:

$$< 0|\frac{1}{2}\int d^3x[\mathbf{E}^2 + \mathbf{B}^2]|0 >= \frac{1}{2}\sum_{\alpha,\mathbf{k}}\omega_{\mathbf{k}} \quad (4.54)$$

H. Casimir (1909-2000), a dutch physicist, wrote the paper on this effect in 1948. This effect was experimentally detected in 1958 by Spaarnay and checked with percent accuracy by Lamoreaux (1998).

We will show that the vacuum energy (4.54) generates an actractive force between the faces of metallic plates at a distance d, such that the force for surface unit

$$f(d) = \frac{F(d)}{L^2} \sim \frac{\hbar c}{d^4} \quad (4.55)$$

Let us consider two metallic squared parallel plates at distance d inside a the cubic box of volume L^3. Let us assume that the conducting plates are orthogonal to the z axis. The electric field in the vacuum is solution of the Maxwell equation with the boundary condition that the tangential field $\mathbf{E_{tg}}$ has to vanish on the conducting wall ($z = 0, d$).

Therefore the tangential field behaves as

$$E_{tg} \sim \sin(k_z z) \tag{4.56}$$

with

$$k_z = \frac{n\pi}{d}, \quad with\ n = 1, 2... \tag{4.57}$$

The energy is

$$\omega_{\mathbf{k}} = \sqrt{k_x^2 + k_y^2 + (\frac{n\pi}{d})^2} \tag{4.58}$$

with

$$k_{x,y} = \frac{n_{x,y}\pi}{L}, \quad n_{x,y} = -\infty, ...\infty \tag{4.59}$$

So the total vacuum energy is

$$U(d) = 2\frac{1}{2}\sum_{k_x,k_y,n} \omega_{\mathbf{k}} \tag{4.60}$$

where the 2 comes from the polarization sum. Let us define $k = \sqrt{k_x^2 + k_y^2}$ so that $kdk = \omega_{\mathbf{k}} d\omega_{\mathbf{k}} \equiv \omega d\omega$ where

$$\omega = \sqrt{k^2 + \left(\frac{n\pi}{d}\right)^2} \tag{4.61}$$

and pass to the continuum (the plaque lenght $L \to \infty$):

$$\begin{aligned} U(d) &= L^2 \sum_{n=1}^{\infty} \int \frac{d^2k}{(2\pi)^2}\omega \\ &= \frac{L^2}{(2\pi)^2}\sum_{n=1}^{\infty}\int_0^{\infty} kdk \int_0^{2\pi} d\varphi\omega \\ &= \frac{L^2}{2\pi}\sum_{n=1}^{\infty}\int_{n\pi/d}^{\infty} d\omega\omega^2 \end{aligned} \tag{4.62}$$

The integral is divergent; there are several ways to regularize the energy to get finite physical results. The result does not depend on the chosen regularization (see for example [4]). Let us consider, by introducing a convergence factor $\exp(-\epsilon\omega)$, the following expression

$$\begin{aligned} U(d,\epsilon) &= \frac{L^2}{2\pi}\sum_{n=1}^{\infty}\int_{n\pi/d}^{\infty} d\omega\omega^2 e^{-\epsilon\omega} \\ &= \frac{L^2}{2\pi}\frac{d^2}{d\epsilon^2}\sum_{n=1}^{\infty}\int_{n\pi/d}^{\infty} d\omega e^{-\epsilon\omega} \\ &= \frac{L^2}{2\pi}\frac{d^2}{d\epsilon^2}\sum_{n=1}^{\infty}\frac{e^{-\epsilon n\pi/d}}{\epsilon} \\ &= \frac{L^2}{2\pi}\frac{d^2}{d\epsilon^2}\Big[\frac{1}{\epsilon}(\frac{1}{1-e^{-\pi\epsilon/d}}-1)\Big] \end{aligned} \tag{4.63}$$

The energy for surface unit is given by

$$\frac{U(d,\epsilon)}{L^2} = \frac{1}{2\pi}\frac{d^2}{d\epsilon^2}\Big[\frac{1}{\epsilon}(\frac{1}{1-e^{-\pi\epsilon/d}}-1)\Big] \tag{4.64}$$

To evaluate the limit for $\epsilon \to 0$ we need the following series

$$\frac{1}{1-e^t} = -\sum_{n=0}^{\infty} B_n \frac{t^{n-1}}{n!} \tag{4.65}$$

where B_n are the Bernouilli numbers ($B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$). The Bernouilli numbers can be evaluated by expanding in Taylor series

$$\frac{z}{1-e^z} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \tag{4.66}$$

Therefore we obtain

$$\begin{aligned} \frac{U(d,\epsilon)}{L^2} &= -\frac{1}{2\pi}\frac{d^2}{d\epsilon^2}\Big[\frac{1}{\epsilon}[1+\sum_{n=0}^{\infty} B_n\frac{(-\pi\epsilon/d)^{n-1}}{n!}\Big] \\ &= -\frac{1}{2\pi}\frac{d^2}{d\epsilon^2}\Big[\frac{1}{\epsilon}[1 - B_0\frac{d}{\pi\epsilon} + B_1 - B_2\frac{\pi\epsilon}{2d} - B_4\frac{\pi^3\epsilon^3}{24d^3} + \mathcal{O}(\epsilon^4)]\Big] \end{aligned} \tag{4.67}$$

Neglecting terms that do not contribute for $\epsilon \to 0$ we get

$$\begin{aligned} \frac{U(d,\epsilon)}{L^2} &= 3\frac{B_0 d}{\pi^2\epsilon^4} - (1+B_1)\frac{1}{\pi\epsilon^3} + \frac{B_4\pi^2}{24d^3} \\ &= C_0 d + C_1 + \frac{C_2}{d^3} \end{aligned} \tag{4.68}$$

In conclusion we have the C_2 finite term which gives the correct result

$$f(d) = -\frac{\partial}{\partial d}\frac{U(d,\epsilon)}{L^2} = -\frac{\pi^2}{240d^4} \tag{4.69}$$

but still two divergent terms, C_0, C_1 when $\epsilon \to 0$. The second C_1 does not contribute to the force, so we can put $C_1 = 0$. The second can be treated with a trick, which consists in introducing two additional external plates at distance $2D$ so that we end with three condensators. The total energy for surface unit is now

$$\begin{aligned} \frac{U(d,D,\epsilon)}{L^2} &= 2[C_0(D-d/2) + \frac{C_2}{(D-d/2)^3}] + C_0 d + \frac{C_2}{d^3} \\ &= 2C_0 D + C_2(\frac{1}{d^3} + \frac{2}{(D-d/2)^3}) \end{aligned} \tag{4.70}$$

Then

$$f(d,D) = -\frac{\partial}{\partial d}\frac{U(d,D,\epsilon)}{L^2} = -\frac{3C_2}{d^4} + C_2(-3)\frac{1}{(D-d/2)^4}(-1/2) \tag{4.71}$$

and taking the limit $D \to \infty$

$$\lim_{D\to\infty} f(d,D) = -\frac{\pi^2}{240d^4} \tag{4.72}$$

We have obtained the result in the system of natural units. The dimensions of the force for unit surface f are L^{-4}. In the cgs units we have $[f] = ML^{-1}T^{-2}$. On the other hand

$$[\hbar c] = ML^3T^{-2} \tag{4.73}$$

Therefore to get the right dimensions in the cgs system, we have to multiply by $\hbar c$

$$\frac{F}{A} = -\frac{\pi^2}{240}\frac{\hbar c}{d^4} \tag{4.74}$$

For $d = 10\mu m$, we obtain a tiny force

$$\frac{F}{A} = -1.3 \times 10^{-6} dyne/cm^2 \tag{4.75}$$

The gravitational force for unit area of two plates of $m = 1g$ and $L = 1cm$ at a distance of $10\mu m$ is comparable. However at smaller distances the Casimir force becomes dominant.

5 Hamiltonian for a system of non relativistic charged particles interacting through the electromagnetic field

In this chapter we consider several processes of interaction between matter and the electromagnetic field, like emission and absorption of photons by atoms, the scattering of photons over atoms and the emission of light by charged particles (Cherenkov effect[6]). In order to compute these processes, we need to consider a system of non relativistic charged particles interacting through the electromagnetic field.

The Lagrangian for a system of N non relativistic particles with mass m_r and charge e_r in interaction between themselves through the electromagnetic field is given by

$$L = \sum_r \left[\frac{1}{2} m_r \dot{\boldsymbol{\xi}}_r(t)^2 - e_r A_0(t, \boldsymbol{\xi}_r) + e_r \dot{\boldsymbol{\xi}}_r(t) \cdot \mathbf{A}(t, \boldsymbol{\xi}_r) \right] + \int d^3x \mathcal{L} \tag{5.1}$$

where $\boldsymbol{\xi}_r(t)$ denotes the position of the r-th particle at time t and

$$\mathcal{L} = \frac{1}{2}(\mathbf{E^2} - \mathbf{B^2}) \tag{5.2}$$

Let us perform the Lagrange transform. Defining

$$\begin{aligned} \mathbf{p}_r &= \frac{\partial L}{\partial \dot{\boldsymbol{\xi}}_r} = m_r \dot{\boldsymbol{\xi}}_r + e_r \mathbf{A}(t, \boldsymbol{\xi}_r) \\ \Pi_i &= \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = -E^i = \partial_i A_0 + \dot{A}^i \end{aligned} \tag{5.3}$$

[6]P. Cherenkov (1904-1990), Nobel prize in Physics in 1958

the Hamiltonian is

$$H = \sum_r \mathbf{p}_r(t) \cdot \dot{\boldsymbol{\xi}}_r(t) + \int d^3x \boldsymbol{\Pi} \cdot \dot{\mathbf{A}}(t,\mathbf{x}) - L \tag{5.4}$$

We obtain (from now on $\boldsymbol{\xi}_r \equiv \boldsymbol{\xi}_r(t), \mathbf{p}_r \equiv \mathbf{p}_r(t)$)

$$\begin{aligned} H &= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 + \sum_r e_r A_0(t,\boldsymbol{\xi}_r) \\ &+ \int d^3x \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \int d^3x \Pi_i \partial_i A_0(t,\mathbf{x}) \\ &= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 + \sum_r e_r A_0(t,\boldsymbol{\xi}_r) \\ &+ \int d^3x \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + \int d^3x \partial_i \Pi_i A_0(t,\mathbf{x}) \end{aligned} \tag{5.5}$$

where we have integrated by parts. On the other hand

$$\partial_i \Pi_i = -\partial_i E^i = -\rho(t,\mathbf{x}) = -\sum_r e_r \delta(\mathbf{x} - \boldsymbol{\xi}_r) \tag{5.6}$$

So

$$\begin{aligned} H &= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 + \sum_r e_r A_0(t,\boldsymbol{\xi}_r) \\ &+ \int d^3x \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) - \sum_r e_r \int d^3x \delta(\mathbf{x} - \boldsymbol{\xi}_r) A_0(t,\mathbf{x}) \\ &= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 + \int d^3x \frac{1}{2}(\mathbf{E}^{\mathbf{2}} + \mathbf{B}^{\mathbf{2}}) \\ &= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 \\ &+ \int d^3x \frac{1}{2}[(\dot{\mathbf{A}}(t,\mathbf{x})^2 + (\nabla A_0(t,\mathbf{x}))^2 + 2\dot{\mathbf{A}}(t,\mathbf{x}) \cdot \nabla A_0(t,\mathbf{x}) + (\nabla \times \mathbf{A}(t,\mathbf{x}))^2] \\ &= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 \\ &+ \int d^3x \frac{1}{2}[(\dot{\mathbf{A}}(t,\mathbf{x})^2 - (\nabla^2 A_0(t,\mathbf{x}) A_0(t,\mathbf{x})) + (\nabla \times \mathbf{A}(t,\mathbf{x}))^2] \end{aligned}$$

$$
\begin{aligned}
&= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 \\
&+ \int d^3x \frac{1}{2}[(\dot{\mathbf{A}}(t,\mathbf{x})^2 + \sum_r e_r\delta(\mathbf{x}-\boldsymbol{\xi}_r)A_0(t,\mathbf{x}) + (\nabla\times\mathbf{A}(t,\mathbf{x}))^2] \\
&= \sum_r \frac{1}{2m_r}(\mathbf{p}_r - e_r\mathbf{A}(t,\boldsymbol{\xi}_r))^2 + \frac{1}{2}\sum_{r\neq s}\frac{e_r e_s}{4\pi|\boldsymbol{\xi}_r-\boldsymbol{\xi}_s|} \\
&+ \int d^3x \frac{1}{2}[(\dot{\mathbf{A}}(t,\mathbf{x})^2 + (\nabla\times\mathbf{A}(t,\mathbf{x}))^2]
\end{aligned}
\tag{5.7}
$$

where we have used

$$A_0(\dot{\boldsymbol{\xi}}_r, t) = \sum_{s\neq r}\frac{e_s}{4\pi|\boldsymbol{\xi}_r-\boldsymbol{\xi}_s|} \tag{5.8}$$

In conclusion the Hamiltonian is given by

$$H = H_{atom} + H_{rad} + V \tag{5.9}$$

where

$$H_{atom} = \sum_r \frac{\mathbf{p}_r^2}{2m_r} + V_{coul} \tag{5.10}$$

is the atomic Hamiltonian with

$$V_{coul} = \frac{1}{2}\sum_{r\neq s}\frac{e_r e_s}{4\pi|\boldsymbol{\xi}_r-\boldsymbol{\xi}_s|} \tag{5.11}$$

and

$$
\begin{aligned}
V &= -\sum_r \frac{e_r}{m_r}\mathbf{p}_r\cdot\mathbf{A}(t,\boldsymbol{\xi}_r) + \sum_r \frac{{e_r}^2}{2m_r}(\mathbf{A}(t,\boldsymbol{\xi}_r))^2 \\
&\equiv V_1 + V_2
\end{aligned}
\tag{5.12}
$$

V is the interaction between the charged particles and the electromagnetic field. H_{rad} denotes the Hamiltonian of the electromagnetic field.

Then we perform the first quantization of the atomic system and the quantization of the electromagnetic field by using the standard commutation relations:

$$[\xi_r^i, p_s^j] = i\delta^{ij}\delta^{rs} \tag{5.13}$$

$$[A^i(t,\mathbf{x}), \dot{A}^j(t,\mathbf{y})] = i\left[\delta^{ij}\delta^3(x-y) + \partial^i\partial^j\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right] \tag{5.14}$$

or remembering the expansion in normal modes by using the creation and annihilation operator commutators (4.27),(4.28).

We are now able to compute emission and absortion from an atom by considering the perturbation theory that we recall in the following section and the interaction Hamiltonian given in eq.(5.12). Notice that the general form has a first term linear in $a_{\mathbf{k}}^{\alpha}$ and $a_{\mathbf{k}}^{\alpha\dagger}$, which can describe processes of emission and absorption of one photon. The second which is bilinear and describes processes where the number of photons can change of two or zero photons.

Other processes could be studied like Thomson, Rayleigh, Raman scattering, photoelectric effect, bremstrahlung,..

6 Scattering theory

6.1 S matrix

As we have seen as we progress from the discussion of the previous section of the free fields and particles to the more realistic case of field and particles in interaction, it is much more difficult to find exact solutions to the problem. In these cases, like radiative transitions in atoms, processes of quantum electrodynamics, the solutions can be found only perturbatively, that is by expanding in power of the interaction strength. In electrodynamics the expansion parameter is the fine structure constant $\alpha = e^2/4\pi \sim 1/137$, which is sufficiently small to make succesfull the perturbation series. In the theory of strong interactions, Quantum ChromoDynamics (QCD), the corresponding parameter $\alpha_s \sim 0.1$ and therefore the expansion is more problematic.

To develop the perturbative methods it is first convenient to introduce the *interaction (or Dirac) representation.* Let us start by recalling the Schrödinger equation and representation

$$i\frac{d}{dt}|\phi(t)>_S = H|\phi(t)>_S \tag{6.1}$$

If H does not depend explicitly on the time, we can build the evolution operator

$$U_S(t, t_0) = e^{-iH(t-t_0)} \tag{6.2}$$

such that

$$|\phi(t) >_S = U_S(t, t_0)|\phi(t_0) >_S \tag{6.3}$$

The operator U_S is unitary, $U_S^\dagger U_S = I$.

In the Heisenberg representation operators and states coincide, at $t = t_0$, with the corresponding operators and states of the Schrödinger representation. For general t, if O^S denotes the operator in the Schrödinger representation, the corresponding operator in the Heisenberg representation is

$$O^H = U_S^\dagger O^S U_S \tag{6.4}$$

and satisfies

$$i\frac{d}{dt}O^H = [O^H, H] \tag{6.5}$$

Let us now assume that the Hamiltonian is given by

$$H = H_0 + H_I \tag{6.6}$$

where H_0 is the Hamiltonian in absence of the interaction and H_I is the interaction Hamiltonian. For example in the radiation matter interaction

$$H_I = -\sum_r \frac{e_r}{m_r}\mathbf{p}_r \cdot \mathbf{A}(t, \boldsymbol{\xi}_r) + \sum_r \frac{{e_r}^2}{2m_r}[\mathbf{A}(t, \boldsymbol{\xi}_r)]^2 \tag{6.7}$$

Let us define the vectors in the interaction representation as

$$|\phi(t) >_I = U_0^\dagger|\phi(t) >_S \tag{6.8}$$

with

$$U_0 = e^{-iH_0 t} \tag{6.9}$$

and the operators as

$$O^I = U_0^\dagger O^S U_0 \tag{6.10}$$

The operators satisfy the equation

$$i\frac{d}{dt}O^I = [O^I, H_0] \tag{6.11}$$

while the states

$$i\frac{d}{dt}|\phi(t)>_I = H_I^I(t)|\phi(t)>_I \tag{6.12}$$

where $H_I^I(t)$ is given by

$$H_I^I(t) = U_0^\dagger H_I U_0 \tag{6.13}$$

and represents the Hamiltonian in the interaction representation. In fact

$$\begin{aligned} i\frac{d}{dt}|\phi(t)>_I &= i\frac{d}{dt}U_0^\dagger|\phi(t)>_S = -H_0U_0^\dagger|\phi(t)>_S + U_0^\dagger(H_0+H_I)|\phi(t)>_S \\ &= U_0^\dagger H_I|\phi(t)>_S = H_I^I|\phi(t)>_I \end{aligned} \tag{6.14}$$

Therefore when the interaction is switched off, the state vector remain constant in time. Let us now study the evolution operator in the interaction representation, defining

$$|\phi(t)>_I = U_I(t,t_0)|\phi(t_0)>_I \tag{6.15}$$

with $U_I(t_0,t_0) = I$. Using the equation (6.14), we obtain

$$i\frac{d}{dt}U_I(t,t_0) = H_I^I U_I(t,t_0) \tag{6.16}$$

One of the advantages of the interaction representation is that, when the interaction is turned off, the vectors are constant in time. Usually the interaction is localized in time and one assumes that in the far past and in the far future the states are eigenstates of H_0.

So let us assume that, at the time $t = t_i = -\infty$, the state is described by the vector

$$|\phi(-\infty)> \equiv |i> \equiv \lim_{t\to-\infty}|\phi(t)>_I \tag{6.17}$$

which is an eigenstate of the H_0 Hamiltonian. The S matrix is defined as

$$|\phi(+\infty)> = \lim_{t\to\infty}|\phi(t)>_I = \lim_{t\to\infty, t_0\to-\infty}U_I(t,t_0)|\phi(t_0)> \equiv S|\phi(-\infty)> \tag{6.18}$$

or

$$S = \lim_{t\to\infty, t_0\to-\infty}U_I(t,t_0) \tag{6.19}$$

For any final state $|f>$, eigenstate of H_0, one considers the matrix element

$$< f|\phi(+\infty) >= S_{fi} \tag{6.20}$$

The solution of eq.(6.16) can be obtained in iterative way

$$U_I(t) = I - i\int_{t_0}^{t} H_I^I(t_1)dt_1 + (-i)^2\int_{t_0}^{t} H_I^I(t_1)dt_1\int_{t_0}^{t_1} H_I^I(t_2)dt_2\ldots \tag{6.21}$$

The series of the S matrix is given by the so-called Dyson series

$$S = \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{t_1}dt_2\ldots\int_{-\infty}^{t_{n-1}}dt_n H_I^I(t_1)H_I^I(t_2)\ldots H_I^I(t_n) \tag{6.22}$$

In general $[H_I^I(t_i), H_I^I(t_j)] \neq 0$. Note also that the integrals are time ordered, $t > t_1 > t_2\cdots$. [7]

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[7] We can rewrite (6.22) as

$$S = \sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\ldots\int_{-\infty}^{\infty}dt_n T(H_I^I(t_1)H_I^I(t_2)\ldots H_I^I(t_n)) \tag{6.23}$$

with

$$T(H_I^I(t_1)) = H_I^I(t_1) \tag{6.24}$$

$$T(H_I^I(t_1)H_I^I(t_2)) = \theta(t_1-t_2)H_I^I(t_1)H_I^I(t_2) + \theta(t_2-t_1)H_I^I(t_2)H_I^I(t_1) \tag{6.25}$$

and so on.

Let us check for example the second order term, by considering

$$\begin{aligned}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2 T(H_I^I(t_1)H_I^I(t_2)) &= \int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2 H_I^I(t_1)H_I^I(t_2)\\ &+ \int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2 H_I^I(t_2)H_I^I(t_1)\end{aligned} \tag{6.26}$$

The integral for the left-hand side is over the square $(t_0,t)\times(t_0,t)$. In the first integral of right-hand side is over the triangle white $(t_1 > t_2)$ while the second term is integrated over the triangle $(t_2 > t_1)$. However

$$\begin{aligned}\int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2 H_I^I(t_2)H_I^I(t_1) &= \int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1 H_I^I(t_2)H_I^I(t_1)\\ &= \int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2 H_I^I(t_1)H_I^I(t_2)\end{aligned} \tag{6.27}$$

One usually defines the transition matrix T

$$S_{fi} = \delta_{fi} - 2\pi i \delta(E_f - E_i) T_{fi} \tag{6.30}$$

and a perturbative series for S and T

$$S_{fi} = \delta_{fi} + S^{(1)}_{fi} + S^{(2)}_{fi} + \ldots \tag{6.31}$$

$$T_{fi} = T^{(1)}_{fi} + T^{(2)}_{fi} + \ldots \tag{6.32}$$

The relation with the evolution operator in the Schrödinger representation is given by

$$U_I(t, t_0) = e^{iH_0 t} U_S(t, t_0) e^{-iH_0 t_0} \tag{6.33}$$

The S matrix is unitary

$$S^\dagger S = I \tag{6.34}$$

This is equivalent to the requirement of the conservation of probability.

$$\sum_f < i|S^\dagger|f >< f|S|i >= 1 \tag{6.35}$$

In the application to scattering problems it is convenient sometime to turn on and off the interaction adiabatically to avoid problems with the oscillatory behaviour at $t \to \pm\infty$. This is obtained by replacing the Hamiltonian with

$$H_I \to H_I e^{-\epsilon|t|} \tag{6.36}$$

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so that

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 T(H^I_I(t_1) H^I_I(t_2)) = 2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H^I_I(t_1) H^I_I(t_2) \tag{6.28}$$

Dyson series can be written in a covariant form

$$S = \sum_{n=0} (-i)^n \frac{1}{n!} \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n T(\mathcal{H}^I_I(x_1)\mathcal{H}^I_I(x_2)\ldots\mathcal{H}^I_I(x_n)) \tag{6.29}$$

The only source of non covariance of eq.(6.29) comes from the presence of the T ordering. However $t_1 - t_2 > 0$ is a property which remains true in every Lorentz frame when $(x_1 - x_2)^2$ is time-like while for space-like distances $(x_1 - x_2)^2 < 0$ $[H(x_1), H(x_2)] = 0$ and so the ordering is not relevant.

so that the interaction acts for a time approximately of order $2/\epsilon$. Then at the end of the calculations, after all the integrations have been performed, we take the limit $\epsilon \to 0$.

To first order

$$-2\pi i \delta(E_f - E_i) T_{fi}^{(1)} = S_{fi}^{(1)} = -i \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt' < f|H_I^I|i > \tag{6.37}$$

Recalling we have

$$\begin{aligned}
-2\pi i \delta(E_f - E_i) T_{fi}^{(1)} &= -i \lim_{t_0 \to -\infty, t \to \infty} \int_{t_0}^{t} dt' < f|e^{iH_0 t'} H_I e^{-iH_0 t'}|i > \\
&= -i \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dt' < f|e^{iH_0 t'} e^{-|\epsilon| t'} V e^{-iH_0 t'}|i > \\
&= -i \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dt' e^{i(E_f - E_i)t'} e^{-|\epsilon| t'} < f|V|i > \\
&= -i < f|V|i > \lim_{\epsilon \to 0^+} [\frac{1}{i}(\frac{1}{E_f - E_i - i\epsilon}) - \frac{1}{i}(\frac{1}{E_f - E_i + i\epsilon})] \\
&= - < f|V|i > [Pv\frac{1}{E_f - E_i} + i\pi\delta(E_f - E_i) \\
&\quad -Pv\frac{1}{E_f - E_i} + i\pi\delta(E_f - E_i)] \\
&= -2\pi i \delta(E_f - E_i) < f|V|i >
\end{aligned} \tag{6.38}$$

where we have used, see Appendix **??**,

$$\frac{1}{E_f - E_i \mp i\epsilon} = Pv\frac{1}{E_f - E_i} \pm i\pi\delta(E_f - E_i) \tag{6.39}$$

The same result can be obtained without the adiabatic approximation but working with distributions.

In conclusion

$$T_{fi}^{(1)} =< f|V|i > \tag{6.40}$$

where $< f|V|i >$ is calculated in the Schrödinger representation (at $t = 0$).

The second order result is given by

$$T_{fi}^{(2)} = \sum_n \frac{< f|V|n >< n|V|i >}{E_i - E_n + i\epsilon} \tag{6.41}$$

where $|n>$ denotes the general state.

We have (defining $\tau = t_1 - t_2)$)

$$\begin{aligned}
-2\pi i\delta(E_f - E_i)T_{fi}^{(2)} &= (-i)^2 < f| \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 H_I^I(t_1) H_I^I(t_2)|i> \\
&= -\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 e^{iE_f t_1} \\
&\quad < f|H_0 e^{-iH_0 t_1} e^{+iH_0 t_2} H_I|i> e^{-iE_i t_2} \\
&= -\sum_n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 e^{iE_f t_1} e^{-iE_i t_2} e^{-iE_n(t_1 - t_2)} \\
&\quad < f|H_I|n >< n|H_I|i > \\
&= -\sum_n \int_{-\infty}^{\infty} dt_1 \int_0^{\infty} d\tau e^{iE_f t_1} e^{-iE_n \tau} e^{iE_i(\tau - t_1)} \\
&\quad < f|H_I|n >< n|H_I|i > \\
&= -\sum_n < f|V|n >< n|V|i > 2\pi\delta(E_f - E_i) \\
&\quad \int_0^{\infty} d\tau e^{i(E_i - E_n)\tau} \\
&= \sum_n < f|V|n >< n|V|i > 2\pi \frac{1}{i(E_i - E_n + i\epsilon)}
\end{aligned} \tag{6.42}$$

where use has been made of the Fourier transforms in the distribution space (see Appendix B)

$$\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \theta(x) e^{ipx} dx = -\frac{1}{i\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{1}{p + i\epsilon} \tag{6.43}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{ipx} = \frac{1}{\sqrt{2\pi}} \delta(p) \tag{6.44}$$

For the proof with the adiabatic factor, see for example [5].

6.2 Fermi golden rule

Let us now compute the transition probability by considering the squared modulus of the S matrix element. Proceeding in this way one gets a divergence, generated by $\delta(0)$:

$$2\pi\delta(E_f - E_i)2\pi\delta(E_f - E_i)|T_{fi}|^2 \sim 4\pi^2\delta(0)\delta(E_f - E_i)|T_{fi}|^2 \tag{6.45}$$

The previous step can be made more rigorous by considering the δ distribution as a limit of distributions associated to functions.

It is possible to obtain a finite result, by considering a transition probability per time unit, assuming that the perturbation acts for a finite time T and then taking the limit $T \to \infty$. Let us consider the following representation of δ

$$2\pi\delta(E_f - E_i) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt e^{i(E_f - E_i)t} \tag{6.46}$$

So we have

$$\begin{aligned} \lim_{T\to\infty} \frac{P_{fi}(T)}{T} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{i(E_f - E_i)t} \lim_{T\to\infty} \int_{-T/2}^{T/2} dt e^{i(E_f - E_i)t} |T_{fi}|^2 \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{i(E_f - E_i)t} 2\pi\delta(E_f - E_i)|T_{fi}|^2 \\ &= 2\pi\delta(E_f - E_i)|T_{fi}|^2 \end{aligned} \tag{6.47}$$

In general, when computing the rate of transition in a final state, one adds also a factor taking into account the density of final states or phase space $d\phi_f$. In other words we compute the transition rate times the number of final states

$$dw_{fi} = 2\pi\delta(E_f - E_i)|T_{fi}|^2 d\phi_f \tag{6.48}$$

Eq.6.48 is the *Fermi*[8] *golden rule*. For example, when in the final state we have a particle with momentum k

$$d\phi_f = V^3 \frac{d^3 k_f}{(2\pi)^3} \tag{6.49}$$

To compute the cross section, defined as the transition rate in a group of final states for one scattering center and unit incident flux, we need to calculate

$$d\sigma = \frac{dw_{fi}}{\Phi} \tag{6.50}$$

where Φ is the flux of incoming particles.

[8]E. Fermi, (1901-1954) Nobel prize in Physics in 1938 "for his demonstrations of the existence of new radioactive elements produced by neutron irradiation, and for his related discovery of nuclear reactions brought about by slow neutrons"

7 Radiation processes of first order: emission and absorption of a single photon

We are now ready to compute the emission and the absorption of a photon by an atom. Let us first consider the decay of an atom which emits a photon of momentum $\mathbf{k}$ and polarization α. The initial state is

$$|i> \equiv |A; \dots n_{\mathbf{k},\alpha} \dots > \equiv |A> \otimes | \dots n_{\mathbf{k},\alpha} \dots > \tag{7.1}$$

and the final state

$$|f> \equiv |A'; \dots n_{\mathbf{k},\alpha} + 1 \dots > \equiv |A'> \otimes | \dots n_{\mathbf{k},\alpha} + 1 \dots > \tag{7.2}$$

where $|A(A')>$ denotes the initial (final) atom state. We have to compute the matrix element

$$V_{fi} = < A'; \dots n_{\mathbf{k},\alpha} + 1 \dots |V_1| A; \dots n_{\mathbf{k},\alpha} \dots > \tag{7.3}$$

where V_1 is given in Eq.(5.12). In the coordinate space the initial (final) state is described by the wave function

$$\psi_{A(A')}(\xi_r) = < \xi_1, \cdots \xi_N | A(A') > \tag{7.4}$$

To first order in the perturbation theory we have (neglecting proton contribution with respect to electron one and denoting the electron masses by m)

$$\begin{aligned}
T_{fi}^{(1)} &= -\frac{e}{m} \sum_r < A'; \dots n_{\mathbf{k},\alpha} + 1 \dots |\mathbf{p}_r \cdot \mathbf{A}(0, \xi_r)| A; \dots n_{\mathbf{k},\alpha} \dots > \\
&= -\frac{e}{m} \sum_r \sum_{\mathbf{k}',\alpha'} \frac{1}{\sqrt{2V\omega_{k'}}} < A'; \dots n_{\mathbf{k},\alpha} + 1 \dots |\mathbf{p}_r \cdot \boldsymbol{\epsilon}_{\mathbf{k}'}^{\alpha'} [a_{\mathbf{k}'}^{\alpha'} \exp(i\mathbf{k}' \cdot \boldsymbol{\xi}_r) + h.c.] \\
&\quad |A; \dots n_{\mathbf{k},\alpha} \dots > \\
&= -\frac{e}{m} \sum_r \sum_{\mathbf{k}',\alpha'} \frac{1}{\sqrt{2V\omega_{k'}}} < A'|\mathbf{p}_r \cdot \boldsymbol{\epsilon}_{\mathbf{k}'}^{\alpha'} \exp(-i\mathbf{k}' \cdot \boldsymbol{\xi}_r)|A > \delta_{\alpha,\alpha'} \delta_{\mathbf{k},\mathbf{k}'} \sqrt{n_{\mathbf{k}',\alpha'} + 1} \\
&= -\frac{e}{m} \sum_r \frac{\sqrt{n_{\mathbf{k},\alpha} + 1}}{\sqrt{2V\omega_k}} < A'|\mathbf{p}_r \cdot \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha} \exp(-i\mathbf{k} \cdot \boldsymbol{\xi}_r)|A > \\
&= -\frac{e}{m} \sum_r \frac{\sqrt{n_{\mathbf{k},\alpha} + 1}}{\sqrt{2V\omega_k}} \int d^3\boldsymbol{\xi}_r \psi_{A'}^*(\boldsymbol{\xi}_r) \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha} \cdot (-i\nabla_r)(\exp(-i\mathbf{k} \cdot \boldsymbol{\xi}_r) \psi_A(\boldsymbol{\xi}_r))
\end{aligned} \tag{7.5}$$

If the transitions are in the visible region, $k^{-1} \sim 1000$ Angstrom, then $k\xi_r << 1$ since $\xi_r \sim 1$ Angstrom. So, in the dipole approximation, $\exp(-i\mathbf{k}\cdot\boldsymbol{\xi}_r) \sim 1$,

$$\begin{aligned} T^{(1)}_{fi} &= -\frac{e}{m}\sqrt{\frac{n_{\mathbf{k},\alpha}+1}{2V\omega_k}}\sum_r < A'|\mathbf{p}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}})|A> \\ &= -e\sqrt{\frac{n_{\mathbf{k},\alpha}+1}{2V\omega_k}}\sum_r < A'|\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}})|A> \end{aligned} \quad (7.6)$$

where $\mathbf{v}_r$ is the velocity operator.

Let us now compute the probability of emitting photons of momentum in the interval $\mathbf{k}, \mathbf{k}+\mathbf{dk}$ and polarization α

$$\begin{aligned} dw_{fi} &= 2\pi\delta(E_f - E_i)|T^{(1)}_{fi}|^2\frac{Vd^3k}{(2\pi)^3} \\ &= 2\pi\frac{e^2}{2V\omega_k}(n_{\mathbf{k},\alpha}+1)|\sum_r < A'|\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}(\exp(-i\mathbf{k}\cdot\boldsymbol{\xi}_r))|A>|^2 \\ &\quad \times\delta(E_f - E_i)\frac{Vd^3k}{(2\pi)^3} \\ &= \frac{d^3k}{(2\pi)^2}(n_{\mathbf{k},\alpha}+1)\frac{e^2}{2\omega_k}|\sum_r < A'|\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}(\exp(-i\mathbf{k}\cdot\boldsymbol{\xi}_r))|A>|^2 \\ &\quad \times\delta(E_{A'}+\omega_k - E_A) \\ &= \frac{\alpha_{em}}{2\pi}(n_{\mathbf{k},\alpha}+1)\omega_k d\omega_k d\Omega|\sum_r < A'||\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}(\exp(-i\mathbf{k}\cdot\boldsymbol{\xi}_r))|A>|^2 \\ &\quad \times\delta(E_{A'}+\omega_k - E_A) \end{aligned} \quad (7.7)$$

We can now integrate over ω_k obtaining the probability of emitting a photon in the solid angle $d\Omega$

$$\frac{dw_{fi}}{d\Omega} = \frac{\alpha_{em}}{2\pi}\omega_k(n_{\mathbf{k},\alpha}+1)|\sum_r < A'|\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}(\exp(-i\mathbf{k}\cdot\boldsymbol{\xi}_r))|A>|^2 \quad (7.8)$$

where now $\omega_k = E_A - E_{A'}$. In the dipole approximation

$$\begin{aligned} |\sum_r < A'|\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}(\exp(-i\mathbf{k}\cdot\boldsymbol{\xi}_r))|A>|^2 &\sim |\sum_r < A'|\mathbf{v}_r\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}|A>|^2 \\ &\sim |< A'|\frac{\dot{\mathbf{D}}}{e}\cdot\boldsymbol{\epsilon}^\alpha_{\mathbf{k}}|A>|^2 \end{aligned} \quad (7.9)$$

where we have introduced the dipole operator

$$\mathbf{D} = e \sum_r \xi_r \tag{7.10}$$

Using

$$\dot{\mathbf{D}} = i[H_{atom}, \mathbf{D}] \tag{7.11}$$

we get

$$\dot{\mathbf{D}}_{A'A} = i < A'|[H_{atom}, \mathbf{D}]|A >= i(E_{A'} - E_A)\mathbf{D}_{A'A} \tag{7.12}$$

Therefore we have

$$\frac{dw_{fi}}{d\Omega} = \frac{\alpha_{em}}{2\pi}\omega_k^3(n_{\mathbf{k},\alpha} + 1)|\frac{1}{e}\mathbf{D}_{A'A} \cdot \boldsymbol{\epsilon}_{\mathbf{k}}^\alpha|^2 \tag{7.13}$$

with

$$\frac{1}{e}\mathbf{D}_{A'A} = \int (\Pi_r d^3\boldsymbol{\xi}_r)\psi^*_{A'}(\boldsymbol{\xi}_r) \sum_r \xi_r \psi_A(\boldsymbol{\xi}_r) \tag{7.14}$$

The result is zero also when there is no initial radiation field, $n_{\mathbf{k},\alpha} = 0$. This is the new result which emerges at the quantum level, the so-called spontaneous emission.

When $n_{\mathbf{k},\alpha} = 0$, and we do not observe the polarization of the photon, we can perform some more analytical step:

$$\sum_\alpha \frac{dw_{fi}}{d\Omega} = \frac{\alpha_{em}}{2\pi}\omega_k^3 \sum_\alpha |\frac{1}{e}\mathbf{D}_{A'A} \cdot \boldsymbol{\epsilon}_{\mathbf{k}}^\alpha|^2 \tag{7.15}$$

Since

$$\sum_\alpha |\frac{1}{e}\mathbf{D}_{A'A}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}^\alpha|^2 = \frac{1}{e^2}\sum_\alpha D^{i*}_{A'A}D^j_{A'A}\epsilon^{\alpha i}\epsilon^{\alpha j} = \frac{1}{e^2}D^{i*}_{A'A}D^j_{A'A}(\delta_{ij} - \frac{k_i k_j}{k^2}) = \frac{1}{e^2}|D^\perp_{A'A}|^2 \tag{7.16}$$

In conclusion

$$\sum_\alpha \frac{dw_{fi}}{d\Omega} = \frac{\alpha_{em}}{2\pi}\omega_k^3\frac{1}{e^2}|D^\perp_{A'A}|^2 \tag{7.17}$$

By integrating over all the solid angle, we obtain the life-time τ for the transition $A \to A'$ which is defined by

$$\frac{1}{\tau_{AA'}} = \int d\Omega \sum_\alpha \frac{dw_{fi}}{d\Omega} = \frac{\alpha_{em}}{2\pi}\omega_k^3\frac{1}{e^2}|D_{A'A}|^2(4\pi - \frac{4}{3}\pi) = \frac{4}{3}\omega_k^3\frac{1}{e^2}|D_{A'A}|^2 \tag{7.18}$$

To get the total life time one has to sum over all the possible states in which the atomic state can decay:

$$\frac{1}{\tau_A} = \sum_{A'} \frac{1}{\tau_{AA'}} \tag{7.19}$$

In the cgs system the eq. (7.18) becomes

$$\frac{1}{\tau} = \frac{4}{3}\omega_k^3 \frac{1}{c^2 e^2} |D_{A'A}|^2 \tag{7.20}$$

To get an extimate of the life time for the $2p$ to $1s$ transition for the hydrogen atom, assuming $\omega_k \sim 10^{16}$ sec^{-1}, $D/e \sim 0.5 \times 10^{-8}$ cm, we get $\tau \sim 10^{-8}$ sec.

8 Interaction of the light with the matter

8.1 Scattering Thomson, Rayleigh, Raman

Let us now consider the scattering of the light (photons) by atomic electrons, neglecting the scattering by the protons since interaction Hamiltonian, given in eqs. (6.6) and (6.7), is inversely proportional to m_r. In particular we will compute the scattering of a photon of momentum $\mathbf{k}_1$ and polarization α_1 by an electron bound in a given atom. For simplicity we consider just an atom with one electron. The result can be easily generalized to N electrons. The initial state is

$$|A> \otimes |\mathbf{k}_1, \alpha_1> \tag{8.1}$$

while the final state is

$$|A'> \otimes |\mathbf{k}_2, \alpha_2> \tag{8.2}$$

where k_2 and α_2 are the momentum and the polarization of the final state photon, $|A(A')>$ denote the initial (final) electron state. In conclusion in the process the number of photons does not change $\Delta n = 0$. Since the Hamiltonian V_1 can describe only $\Delta n = \pm 1$ process, we have to go to the next order in the expansion, that is to order (e^2). The Hamiltonian V_2 can describe $\Delta n = 0$ process, since

$$V_2 \sim (a_{\mathbf{k}}^{\alpha} + a_{\mathbf{k}}^{\alpha\dagger})(a_{\mathbf{k}'}^{\alpha'} + a_{\mathbf{k}'}^{\alpha'\dagger}) \tag{8.3}$$

The general amplitude to order e^2 is given by

$$T_{fi}^{(2)} = (V_2)_{fi} + \sum_n \frac{(V_1)_{fn}(V_1)_{ni}}{E_i - E_n + i\epsilon} \tag{8.4}$$

where the sum is over all possible intermediate states $|n> \equiv |N> \otimes |...n_{\mathbf{k},\alpha}...>$,

$$E_i = E_A + \omega_{k_1} \tag{8.5}$$

$E_n = E_N + ...$ denotes the energy of the intermediate state. Let us first consider the case when the energy of the incoming photon is much larger than the level splitting of the atoms. In this case one can neglect the second term of eq.(8.4) and one recovers the classical result of the scattering of the light by a free electron (Thomson[9] scattering). Substituting in V_2 we get

$$V_2 = \frac{e^2}{2m}\sum_{\mathbf{k},\alpha}\sum_{\mathbf{k}',\alpha'} \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}\cdot\boldsymbol{\epsilon}_{\mathbf{k}'}^{\alpha'}\frac{1}{\sqrt{2\omega_k V}\sqrt{2\omega_{k'}V}}(a_{\mathbf{k}}^{\alpha}e^{i\mathbf{k}\cdot\boldsymbol{\xi}} + h.c.)(a_{\mathbf{k}'}^{\alpha'}e^{i\mathbf{k}'\cdot\boldsymbol{\xi}} + h.c.) \tag{8.6}$$

The terms contributing to the final result are $a_{\mathbf{k}}^{\alpha}a_{\mathbf{k}'}^{\alpha'\dagger}$ and $a_{\mathbf{k}}^{\alpha\dagger}a_{\mathbf{k}'}^{\alpha'}$. The final result is

$$\begin{aligned} T_{fi}^{(2)} &= (V_2)_{fi} = \frac{e^2}{m}\epsilon_{\mathbf{k}_1}^{\alpha_1}\cdot\epsilon_{\mathbf{k}_2}^{\alpha_2}\frac{1}{2V\sqrt{\omega_{k_1}\omega_{k_2}}} < A'|e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\xi}}|A> \\ &= \frac{e^2}{m}\epsilon_{\mathbf{k}_1}^{\alpha_1}\cdot\epsilon_{\mathbf{k}_2}^{\alpha_2}\frac{1}{2V\sqrt{\omega_{k_1}\omega_{k_2}}}\int d^3\xi\,\psi_{A'}^*(\xi)e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\xi}}\psi_A(\xi) \end{aligned} \tag{8.7}$$

where we have introduced the wave function of the electron in the state $A(A')$, $\psi_{A(A')}(\xi)$.

The generalization to N electrons is:

$$T_{fi}^{(2)} = \frac{e^2}{m}\epsilon_{\mathbf{k}_1}^{\alpha_1}\cdot\epsilon_{\mathbf{k}_2}^{\alpha_2}\frac{1}{2V\sqrt{\omega_{k_1}\omega_{k_2}}}\sum_r\int \Pi_r d^3\xi_r\psi_{A'}^*(\xi_r)e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\boldsymbol{\xi}_r}\psi_A(\xi_r) \tag{8.8}$$

If the dipole approximation can be used, we get

$$T_{fi}^{(2)} = (V_2)_{fi} = \frac{e^2}{m}\epsilon_{\mathbf{k}_1}^{\alpha_1}\cdot\epsilon_{\mathbf{k}_2}^{\alpha_2}\frac{1}{2V\sqrt{\omega_{k_1}\omega_{k_2}}}\delta_{AA'} \tag{8.9}$$

[9]J.J. Thomson, 1856-1940, Nobel prize in Physics in 1906 for the discovery of the electron

Using the Fermi golden rule we get the probability of scattering per unit time

$$dw = 2\pi\delta(\omega_{k_1} - \omega_{k_2})\frac{e^4}{4\omega_{k_1}{}^2V^2m^2}(\epsilon^{\alpha_1}_{\mathbf{k}_1} \cdot \epsilon^{\alpha_2}_{\mathbf{k}_2})^2\frac{V}{(2\pi)^3}d^3k_2 \tag{8.10}$$

Finally the cross section is obtained by dividing by the incoming flux $\Phi = \rho v = 1/V$, being the photon velocity equal to one in natural units,

$$d\sigma = \frac{dw}{\Phi} = \delta(\omega_{k_1} - \omega_{k_2})(\frac{e^2}{4\pi m\omega_{k_1}}\epsilon^{\alpha_1}_{\mathbf{k}_1} \cdot \epsilon^{\alpha_2}_{\mathbf{k}_2})^2k_2^2dk_2d\Omega_{k_2} \tag{8.11}$$

By integrating over $\omega_{k_2} = k_2$

$$\int_{k_2} d\sigma = r_0^2(\epsilon^{\alpha_1}_{\mathbf{k}_1} \cdot \epsilon^{\alpha_2}_{\mathbf{k}_2})^2d\Omega_{k_2} \tag{8.12}$$

where r_0 is the classical radius of the electron

$$r_0 = \frac{e^2}{4\pi m} \sim 2.8 \text{ fm} = 2.8 \times 10^{-13}\,\text{cm} \tag{8.13}$$

If we do not know the initial polarization and we do not measure the final polarization, we average over the initial polarization and we sum over the final ones

$$\frac{1}{2}\sum_{\alpha_1,\alpha_2} d\sigma = \frac{r_0^2}{2}(1 + \cos^2\theta)d\phi d\cos\theta \tag{8.14}$$

where θ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. In fact we have

$$\begin{aligned}\frac{1}{2}\sum_{\alpha_1,\alpha_2}\epsilon^{\alpha_1 i}_{\mathbf{k}_1}\epsilon^{\alpha_2 i}_{\mathbf{k}_2}\epsilon^{\alpha_1 j}_{\mathbf{k}_1}\epsilon^{\alpha_2 j}_{\mathbf{k}_2} &= \frac{1}{2}\sum_{\alpha_1}\epsilon^{\alpha_1 i}_{\mathbf{k}_1}\epsilon^{\alpha_1 j}_{\mathbf{k}_1}(\delta_{ij} - \frac{k_1^ik_1^j}{k_1^2}) \\ &= \frac{1}{2}(\delta_{ij} - \frac{k_2^ik_2^j}{k_2^2})(\delta_{ij} - \frac{k_1^ik_1^j}{k_1^2}) \\ &= \frac{1}{2}[3 - 1 - 1 + (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2] \\ &= \frac{1}{2}[1 + (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2]\end{aligned} \tag{8.15}$$

By integrating over the angles, we obtain the total Thomson cross section

$$\sigma = \frac{8\pi}{3}r_0^2 \tag{8.16}$$

Using the numerical value of r_0 we get $\sigma = 6.6 \times 10^{-25}\text{cm}^2$.

We can now proceed to compute the cross section in the general case deriving the so called Kramers Heisenberg formula (1925). The possible intermediate states contributing to the second order part of the amplitudes are

$$|N> \otimes |0>, \quad |N> \otimes |k_1, \alpha_1; k_2, \alpha_2> \tag{8.17}$$

that is $|N>$ times the vacuum states and the two photon state. The result for the amplitude is

$$T_{fi}^{(2)} = (V_2)_{fi} + \sum_N [\frac{(V_1^0)_{A'N}(V_1^0)_{NA}}{E_A + \omega_{k_1} - E_N + i\epsilon} + \frac{(V_1^{II})_{A'N}(V_1^{II})_{NA}}{E_A - \omega_{k_2} - E_N + i\epsilon}] \tag{8.18}$$

where

$$(V_1^0)_{A'N} = \frac{e}{m}\frac{1}{\sqrt{2\omega_{k_2}V}} < A'|\mathbf{p}\cdot\epsilon_{\mathbf{k_2}}^{\alpha_2} e^{-i\mathbf{k}_2\cdot\boldsymbol{\xi}}|N> \tag{8.19}$$

$$(V_1^0)_{NA} = \frac{e}{m}\frac{1}{\sqrt{2\omega_{k_1}V}} < N|\mathbf{p}\cdot\epsilon_{\mathbf{k_1}}^{\alpha_1} e^{i\mathbf{k}_1\cdot\boldsymbol{\xi}}|A> \tag{8.20}$$

$$(V_1^{II})_{A'N} = \frac{e}{m}\frac{1}{\sqrt{2\omega_{k_1}V}} < A'|\mathbf{p}\cdot\epsilon_{\mathbf{k_1}}^{\alpha_1} e^{i\mathbf{k}_1\cdot\boldsymbol{\xi}}|N> \tag{8.21}$$

$$(V_1^{II})_{NA} = \frac{e}{m}\frac{1}{\sqrt{2\omega_{k_2}V}} < N|\mathbf{p}\cdot\epsilon_{\mathbf{k_2}}^{\alpha_2} e^{-i\mathbf{k}_2\cdot\boldsymbol{\xi}}|A> \tag{8.22}$$

and $(V_2)_{fi}$ is given in eq.(8.7). In the dipole approximation we get

$$T_{fi}^{(2)} = \frac{1}{2V\sqrt{\omega_{k_1}\omega_{k_2}}} f_{A'A} \tag{8.23}$$

with

$$\begin{aligned} f_{A'A} = \frac{e^2}{m}\epsilon_{\mathbf{k_1}}^{\alpha_1}\cdot\epsilon_{\mathbf{k_2}}^{\alpha_2}\delta_{AA'} + & \frac{e^2}{m^2} \sum_N [\frac{< A'|\mathbf{p}\cdot\epsilon_{\mathbf{k_2}}^{\alpha_2}|N>< N|\mathbf{p}\cdot\epsilon_{\mathbf{k_1}}^{\alpha_1}|A>}{E_A + \omega_{k_1} - E_N + i\epsilon} \\ + & \frac{< A'|\mathbf{p}\cdot\epsilon_{\mathbf{k_1}}^{\alpha_1}|N>< N|\mathbf{p}\cdot\epsilon_{\mathbf{k_2}}^{\alpha_2}|A>}{E_A - \omega_{k_2} - E_N + i\epsilon}] \end{aligned} \tag{8.24}$$

To obtain the cross section we divide by the flux:

$$\frac{d\sigma}{d\Omega} = 2\pi\delta(E_A + \omega_{k_1} - E_{A'} - \omega_{k_2})\frac{1}{4V^2\omega_{k_1}\omega_{k_2}}|f_{A'A}|^2 k_2^2 dk_2 \frac{V}{(2\pi)^3}\frac{1}{\frac{1}{V}} \tag{8.25}$$

By integrating over k_2, we obtain

$$\int \frac{d\sigma}{d\Omega} = \frac{\omega_{k_2}}{\omega_{k_1}} |\frac{f_{A'A}}{4\pi}|^2 \tag{8.26}$$

The case $A' = A$ $\omega_{k_1} = \omega_{k_2}$ correspond to the elastic case (Rayleigh[10] scattering) while the inelastic case, $A' \neq A$ $\omega_{k_1} \neq \omega_{k_2}$, corresponds to the Raman[11] effect.

Note that when the energy of the initial photon $\omega_{k_1} = E_N - E_A$, the amplitude as given in eq.(8.24) diverges and consequently the cross section becomes infinite. Since the cross section is a measurable quantity, this means that the calculation to second order becomes inadequate and higher orders become relevant.

What is happening is that we have assumed the intermediate states N as stable, neglecting their life time due to the instability for the spontaneous emission. If the probability of finding an electron in a generic energy level E_N decreases for spontaneous emission as

$$\exp{(-\frac{t}{\tau_N})} = \exp{(-\Gamma_N t)} \tag{8.27}$$

where τ_N (Γ_N) is the life time (width) of the state N, the corresponding amplitude behaves as

$$\exp{(-\Gamma_N t/2)} \tag{8.28}$$

This is equivalent to a time evolution of the state as

$$\exp{[-i(E_N - i\frac{\Gamma_N}{2})t]} \tag{8.29}$$

In conclusion in presence of a resonant process, when $\omega_{k_1} \sim E_N - E_A$ we can replace

$$\frac{1}{E_A + \omega_{k_1} - E_N} \to \frac{1}{E_A + \omega_{k_1} - E_N + i\frac{\Gamma_N}{2}} \tag{8.30}$$

so that the cross section, close to the resonance, assumes the classic Lorentz form

$$\frac{d\sigma}{d\Omega} \sim \frac{|C|^2}{|E_A + \omega_{k_1} - E_N + i\frac{\Gamma_N}{2}|^2} = \frac{|C|^2}{(E_A + \omega_{k_1} - E_N)^2 + \frac{\Gamma_N^2}{4}} \tag{8.31}$$

[10]J.W.S.Rayleigh (1842-1919), Nobel prize in Physics in 1904 for discovery of Argon

[11]C.V. Raman (1888-1970), Nobel prize in Physics in 1930 "for his work on the scattering of light and for the discovery of the effect named after him"

8.2 Calculation of the total width

In order to compute the total width, we start considering the total shift in the energy of the bound state to order $O(e^2)$:

$$\Delta E_n =< n|H_I|n > + \sum_{m\neq n} \frac{< n|H_I|m >< n|H_I|m >}{E_n - E_m} \tag{8.32}$$

where

$$|n >= |N > \otimes |0 > \tag{8.33}$$

is a pure atom state, no photons are present. The interaction Hamiltonian is

$$H_I = V_1 + V_2 \tag{8.34}$$

with

$$V_1 = -\frac{e}{m}\mathbf{p}\cdot\mathbf{A}(\xi,0), \quad V_2 = \frac{e^2}{2m}(\mathbf{A}(\xi,0))^2 \tag{8.35}$$

Assuming H_I normal ordered

$$< n|H_I|n >=< n|V_2|n >= 0 \tag{8.36}$$

and only the second term of eq. (8.32) is different from zero

$$\Delta E_n = \sum_{m\neq n} \frac{< n|V_1|m >< m|V_1|n >}{E_n - E_m} \tag{8.37}$$

Therefore since $|n >= |N > \otimes|0 >$ the only intermediate state allowed is

$$|m >= |M > \otimes |1_{\mathbf{k},\alpha} > \tag{8.38}$$

and the result is given by the amplitude for the spontaneous emission

$$(< N| \otimes < 0|)V_1(|M > \otimes |1_{\mathbf{k},\alpha} >) = T_{NM}^{spont.emiss.} \tag{8.39}$$

where we can use the result of the spontaneous emission (7.8) for $n_{\mathbf{k},\alpha} = 0$ for one electron or

$$\begin{aligned} T_{MN}^{spont.emiss.} &= -\frac{e}{m}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} < N|\mathbf{p}\cdot\boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha}\exp\left(-i\mathbf{k}\cdot\xi\right)|M > \\ &= -\frac{e}{m}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}|t_{MN}|^2 \end{aligned} \tag{8.40}$$

So we get

$$\begin{aligned}\Delta E_n &= \frac{e^2}{2m^2V}\sum_{\mathbf{k},\alpha}\sum_M \frac{1}{\omega_{\mathbf{k}}}\frac{|t_{MN}|^2}{E_N - E_M - \omega_k} \\ &\to \frac{e^2}{2m^2}\int \frac{d^3k}{(2\pi)^3}\sum_M \frac{1}{\omega_{\mathbf{k}}}\frac{|t_{MN}|^2}{E_N - E_M - \omega_k} \\ &= \frac{\alpha}{m^2}\frac{1}{(2\pi)^2}\int d\omega_k d\Omega \omega_k \sum_M \frac{|t_{MN}|^2}{E_N - E_M - \omega_k}\end{aligned} \tag{8.41}$$

On the other hand using (7.8) for spontaneous emission for atoms with a single electron, we have

$$\begin{aligned}\frac{dw_{NM}}{d\Omega} &= \frac{\alpha}{2\pi m^2}\omega_k| < N|\mathbf{p}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}(\exp{(-i\mathbf{k}\cdot\xi)}|M > |^2 \\ &= \frac{\alpha}{2\pi m^2}\omega_k|t_{MN}|^2\end{aligned} \tag{8.42}$$

where $\omega_k = E_N - E_M$.

In conclusion we can rewrite ΔE_N as

$$\Delta E_n = \frac{1}{2\pi}\sum_M \int_0^\infty d\omega_k \frac{1}{E_N - E_M - \omega_k}\frac{dw_{NM}}{d\Omega} \tag{8.43}$$

where $\frac{dw_{NM}}{d\Omega}$ is the probability of spontaneous emission of a photon of frequency ω_k which assumes any value from zero to infinity.

The integral (8.45) has a pole for $\omega_k = E_N - E_M$. Let us replace the denominator with $E_N - E_M - \omega_k + i\epsilon$, so we have to compute

$$\Delta E_n = \frac{1}{2\pi}\sum_M \int_0^\infty d\omega_k \frac{1}{E_N - E_M - \omega_k + i\epsilon}\frac{dw_{NM}}{d\Omega} \tag{8.44}$$

Using (C.6) we have

$$\Delta E_n = \frac{1}{2\pi}\sum_M [Pv\int_0^\infty d\omega_k \frac{1}{E_N - E_M - \omega_k} - i\pi\int_0^\infty d\omega_k \delta(E_N - E_M - \omega_k)]\frac{dw_{NM}}{d\Omega} \tag{8.45}$$

By taking the imaginary part

$$\begin{aligned} Im[\Delta E_n] &= -\frac{1}{2\pi}\sum_M \int_0^\infty d\omega_k[\pi\delta(E_N - E_M - \omega_k)]\frac{dw_{NM}}{d\Omega} \\ &= -\frac{1}{2}\sum_M \frac{dw_{NM}}{d\Omega}|_{\omega_k = E_N - E_M} \\ &= -\frac{1}{2}\Gamma_N \end{aligned} \quad (8.46)$$

where Γ_N is the total width of the state N due to spontaneous emission. Therefore the time evolution is now

$$\psi_N(t) = e^{-i(E_N + Re[\Delta E_N] - \frac{i}{2}\Gamma_N)t}\psi_N(0) = e^{-i(E_N + Re[\Delta E_N])t}e^{-\frac{1}{2}\Gamma_N t}\psi_N(0) \quad (8.47)$$

and the probability decreases as

$$|\psi_N(t)|^2 = e^{-\Gamma_N t}|\psi_N(0)|^2 \quad (8.48)$$

The real part of ΔE_N gives the shift in the energy of the bound state corresponding to the *Lamb* shift. The integral

$$Re[\Delta E_n] = \frac{1}{2\pi}\sum_M Pv\int_0^\infty d\omega_k[\frac{1}{E_N - E_M - \omega_k}]\frac{dw_{NM}}{d\Omega} \quad (8.49)$$

is divergent and a special procedure of renormalization of the mass of the electron is necessary [20, 17].

9 Cherenkov effect

In 1934, while Pavel Cherenkov (1904-1990) was studying, under the supervision of Sergei Vavilov at the Physical Institute of the USSR Academy of Science, the luminescence of liquids of uranyl salt under irradiation of gamma rays from radium, he discovered a new bleu glow. Vavilov suggested that the effect could be due to bremstrahlung of electrons that were knocked out by the gamma rays of the radium. However the correct explanation, which is not bremstrahlung, was given by Tamm and Frank[12] (1937): they considered

[12] I.M. Franck, I.E. Tamm, Nobel prize in Physics in 1958 with Cherenkov

the field of a point like charged particle moving in a medium uniformly and rectilinearly and show that if the velocity v of the particle is higher than the velocity of the light in the medium c/n, the particle emits a radiation in a cone of angle θ such that

$$\cos\theta = \frac{c}{nv} \tag{9.1}$$

Let us first consider when a free charged particle of momentum p, traveling in a medium of index n, can emit a photon of momentum k. From the conservation of the momentum we get

$$p^\mu = (p' + k)^\mu \tag{9.2}$$

or

$$(p')^2 = m^2 = (p-k)^2 = m^2 - 2p^\mu k_\mu + \omega_k^2 - \mathbf{k}^2 \tag{9.3}$$

We conclude that

$$2(p^0\omega_k - \mathbf{p}\cdot\mathbf{k}) = \omega_k^2 - \mathbf{k}^2 = (\frac{1}{n^2} - 1)k^2 \tag{9.4}$$

Then

$$\mathbf{p}\cdot\mathbf{k} = E\frac{k}{n} + \frac{n^2-1}{2n^2}k^2 \tag{9.5}$$

or

$$\begin{aligned} \cos\theta &= \frac{E}{p}\frac{1}{n} + \frac{n^2-1}{2n^2}\frac{k}{p} \\ &= \frac{1}{nv} + \frac{n^2-1}{2n}\frac{\omega}{m\gamma v} \end{aligned} \tag{9.6}$$

The final relativistic formula for the Cherenkov angle is

$$\cos\theta = \frac{1}{nv}\left[1 + \frac{n^2-1}{2m}\omega\sqrt{1-v^2}\right] \tag{9.7}$$

where we have used $E = m\gamma$, $\mathbf{p} = m\gamma\mathbf{v}$.

Since the Cherenkov light in the water is in the visible light corresponding to 400-700 nm$\sim$ or 10^{-1} eV^{-1} the ratio $k/m \sim 10^{-4}$, we can neglect the second term of eq.(9.7), therefore the angle is approximately

$$\cos\theta = \frac{1}{nv} \tag{9.8}$$

This result implies

$$\frac{1}{nv} \leq 1 \tag{9.9}$$

or $v \geq 1/n$. So in the vacuum this process is forbidden, since v cannot be greater the 1, (the light velocity $c = 1$ in natural unit). However in a medium v can be greater than $1/n$ and the process is allowed.

In conclusion a charged particle, which travels at a velocity that exceeds c/n, can emit a photon. The refractive index of the water between 400-700 nm is in the range 1.33-1.34. So the critical velocity in the water is $v \sim 0.75$.

This effect has been used to build detectors for charged particles. Today the largest Cherenkov detector is the Super-Kamiokande detector, which contains 50000 tons of water, 11200 photomultipliers, and is located in Japan. This experiment has discovered neutrino oscillations. The Cherenkov light is emitted in a cone around the direction of a charged particle. The photomultiplier tubes of the tank detect this Cherenkov light and give information of the quantity of the detected light and the timing of the detection. They give information also on the energy, direction, interaction point and type of the charged particle.

We will derive the Cherenkov emission of a charged particle by considering the quantized electromagnetic field in a medium. For the total energy associated to the electromagnetic field see [6, 7]. In order to get the standard form for the Hamiltonian in terms of creation and annihilation operators

$$H = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \omega_k (a_{\mathbf{k}}^{\alpha\dagger} a_{\mathbf{k}}^{\alpha} + \frac{1}{2}) \tag{9.10}$$

we need to include the refraction index in the expansion [6, 7]

$$\mathbf{A}(x) = \sum_{\mathbf{k}} \sum_{\alpha=1,2} \frac{1}{n\sqrt{2V\omega_k}} \boldsymbol{\epsilon}_{\mathbf{k}}^{\alpha} [a_{\mathbf{k}}^{\alpha} e^{-ikx} + h.c.], \quad k^i = n_i \frac{2\pi}{L} \tag{9.11}$$

where now $k^0 \equiv \omega_k = k/n$ and $n = n(\omega_k)$. To calculate the probability that an electron emits Cherenkov radiation, for simplicity of the calculations we will consider the non relativistic approximation for the electron. The initial state is

$$|i\rangle = |\mathbf{p}\rangle \otimes |0\rangle \tag{9.12}$$

where $|\mathbf{p}\rangle$ represent a non relativistic electron state of momentum $\mathbf{p}$ and $|0\rangle$ the state with no photons. The final state is

$$|f\rangle = |\mathbf{p}'\rangle \otimes |1_{\mathbf{k},\alpha}\rangle \tag{9.13}$$

where $|1_{\mathbf{k},\alpha}\rangle$ represent the Cherenkov photon. Let us compute

$$\begin{aligned}
V_{fi} = \langle f|V|i\rangle &= \langle \mathbf{p}'| \otimes \langle 1_{\mathbf{k},\alpha}|(-e\frac{\mathbf{p}}{m} \cdot \sum_{\mathbf{k}'} \sum_{\alpha'=1,2} \frac{1}{n\sqrt{2V\omega_{k'}}} \boldsymbol{\epsilon}^{\alpha'}_{\mathbf{k}'}[a^{\alpha'}_{\mathbf{k}'} e^{i\mathbf{k}'\cdot\mathbf{x}} + h.c.])|\mathbf{p}\rangle \otimes |0\rangle \\
&= \langle \mathbf{p}'|(-e\frac{\mathbf{p}}{m} \cdot \boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}} \frac{1}{n\sqrt{2V\omega_k}} e^{-i\mathbf{k}\cdot\mathbf{x}})|\mathbf{p}\rangle \\
&= -\frac{e\mathbf{p}'\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}}{nm\sqrt{2V\omega_k}} \langle \mathbf{p}'|e^{-i\mathbf{k}\cdot\mathbf{x}})|\mathbf{p}\rangle \\
&= -\frac{e\mathbf{p}'\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}}{nm\sqrt{2V\omega_k}} \delta_{\mathbf{p}'-\mathbf{p}+\mathbf{k},0} \\
&= -\frac{e\mathbf{p}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}}{nm\sqrt{2V\omega_k}} \delta_{\mathbf{p}'+\mathbf{k}-\mathbf{p},0}
\end{aligned} \tag{9.14}$$

where we have used the non relativistic wave functions

$$< x|p >= \frac{1}{\sqrt{V}} e^{i\mathbf{p}\cdot\mathbf{x}} \tag{9.15}$$

Then one can compute the transition rate that an electron of momentum $\mathbf{p}$ emits a photon of momentum $\mathbf{k}$ times the number of final states. Passing to the continuum, we get

$$\begin{aligned}
dw &= 2\pi\delta(E_f - E_i)|V_{fi}|^2 \frac{V}{(2\pi)^3} d^3p' \frac{V}{(2\pi)^3} d^3k \\
&= 2\pi\delta(E_f - E_i)(\frac{e\mathbf{p}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}}{nm\sqrt{2V\omega_k}})^2 \frac{(2\pi)^3}{V} \delta^3(p' + k - p) \frac{V}{(2\pi)^3} d^3p' \frac{V}{(2\pi)^3} d^3k \\
&= \frac{\pi e^2}{n^2\omega_k} |\frac{\mathbf{p}}{m}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}|^2 \frac{d^3p'}{(2\pi)^3} d^3k \delta^3(p' + k - p)\delta(E_{p'} + \omega_k - E_p)
\end{aligned} \tag{9.16}$$

Integrating over d^3p'

$$\begin{aligned}
\int_{p'} dw &= \frac{\pi e^2}{n^2\omega_k} |\frac{\mathbf{p}}{m}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}|^2 \frac{1}{(2\pi)^3} d^3k \delta(E_{p'} + \omega_k - E_p) \\
&\sim \frac{\pi e^2}{n^2\omega_k} |\frac{\mathbf{p}}{m}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}|^2 \frac{1}{(2\pi)^3} d^3k \delta(\boldsymbol{v} \cdot \mathbf{k} - \omega_k) \\
&= \frac{e^2}{8\pi^2} |\frac{\mathbf{p}}{m}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}|^2 n\omega_k d\omega_k d\cos\theta d\phi \delta(\mathbf{v} \cdot \mathbf{k} - \omega_k)
\end{aligned} \tag{9.17}$$

where we have used

$$E_{p'} = E_{p-k} \sim E_p - \frac{\partial E}{\partial \mathbf{p}} \cdot \mathbf{k} \tag{9.18}$$

However

$$\delta(\boldsymbol{v} \cdot \mathbf{k} - \omega_k) = \frac{1}{vk}\delta(\cos\theta - \frac{1}{nv}) \tag{9.19}$$

Therefore we get, where using the non relativistic relation for the momentum $\mathbf{p}$,

$$\begin{aligned} \int_{p'} dw &= \frac{e^2}{8\pi^2}|\boldsymbol{v}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}|^2 n\omega_k d\omega_k d\cos\theta d\phi \frac{1}{vk}\delta(\cos\theta - \frac{1}{nv}) \\ &= \frac{e^2}{8\pi^2}|\boldsymbol{v}\cdot\boldsymbol{\epsilon}^{\alpha}_{\mathbf{k}}|^2 d\omega_k d\phi d\cos\theta \frac{1}{v}\delta(\cos\theta - \frac{1}{nv}) \end{aligned} \tag{9.20}$$

Summing over the polarization and finally integrating over ϕ and θ

$$\begin{aligned} \int_{\phi,\cos\theta}\sum_{\alpha}\int_{p'} dw &= \int_{\phi,\cos\theta}\sum_{\alpha}\frac{e^2}{8\pi^2}v^i v^j \epsilon^{\alpha i}\epsilon^{\alpha j}\frac{1}{v}\delta(\cos\theta - \frac{1}{nv})d\omega_k \\ &= \int_{\phi,\cos\theta}\frac{e^2}{8\pi^2}v^i v^j(\delta^{ij} - \frac{k^i k^j}{k^2})\frac{1}{v}\delta(\cos\theta - \frac{1}{nv})d\omega_k \\ &= \frac{e^2}{4\pi}v(1 - \frac{1}{n^2v^2})d\omega_k \\ &= \alpha v(1 - \frac{1}{n^2v^2})d\omega_k \end{aligned} \tag{9.21}$$

This result, that we have obtained using a non relativistic approach, coincides with the calculation performed using as initial and final states the electron described by the Dirac field and the Hamiltonian interaction given by eq.(10.112) of next Section, in the limit $\omega_k/E_p << 1$.

The energy loss of the electron per unit lenght of the trajectory is

$$\frac{dE}{dx} = \frac{1}{v}\frac{dE}{dt} = \int_0^{\omega_{max}}\frac{1}{v}\alpha v(1 - \frac{1}{n^2v^2})\omega_k d\omega_k = \alpha\int_0^{\omega_{max}}(1 - \frac{1}{n^2v^2})\omega_k d\omega_k \tag{9.22}$$

The integral is cutoff by ω_{max},

$$\omega \leq \omega_{max} \sim \frac{(nv-1)2m}{n^2-1}\frac{\gamma}{v} \tag{9.23}$$

as we obtain by requiring $\cos\theta \leq 1$ from eq.(9.7).

10 The Dirac field

10.1 The Dirac equation: classical theory

Let us now consider the classical theory of the Dirac equation (1928), which describes relativistic particles with spin $\frac{1}{2}$. Let us first write the following first order differential equation:

$$i\frac{\partial\psi(x)}{\partial t} = [\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}) + \beta m]\psi(x) \tag{10.1}$$

where α^i, β are four hermitian matrices $n\times n$ and ψ a vector of dimension n. By iterating $i\frac{\partial}{\partial t}$ we obtain

$$\begin{aligned}
-\frac{\partial^2\psi(x)}{\partial t^2} &= [\boldsymbol{\alpha}\cdot(-i\boldsymbol{\nabla}) + \beta m]^2\psi \\
&= [-i\alpha^i\partial^i + \beta m][-i\alpha^j\partial^j + \beta m]\psi \\
&= [-\alpha^i\alpha^j\partial^i\partial^j - i(\alpha^i\beta + \beta\alpha^i)\partial^i m + \beta^2 m^2]\psi \\
&= [-\frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial^i\partial^j - i(\alpha^i\beta + \beta\alpha^i)\partial^i m + \beta^2 m^2]\psi
\end{aligned} \tag{10.2}$$

By requiring the validity of Klein Gordon equation one gets

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}, \;\; \alpha^i\beta + \beta\alpha^i = 0, \;\; \beta^2 = 1 \tag{10.3}$$

or

$$[\alpha^i, \alpha^j]_+ = \delta^{ij}, [\beta, \alpha^i]_+ = 0, \quad \beta^2 = 1, \tag{10.4}$$

where $[A, B]_+ \equiv AB + BA$ is the anti-commutator. The minimal dimension n of the matrices where the previous relations hold is four. In the Dirac-Pauli representation they are represented as

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \;\; \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{10.5}$$

The equation can be rewritten in the form

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \tag{10.6}$$

where

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i \tag{10.7}$$

are Dirac 4×4 matrices satisfying

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} \tag{10.8}$$

and

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{10.9}$$

In the Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{10.10}$$

Therefore $\psi(x)$ is a four component wave function. Usually the components are indicated by $\psi_\alpha(x)$, $\alpha = 1, 2, 3, 4$.

10.2 Lorentz and parity transformation

Let us now assume that the Dirac equation be valid in any inertial frame. Then in the S' system, where $x' = \Lambda x$, assuming the relativity principle, Dirac equation must have the same form

$$(i\gamma^{\mu\prime}\partial'_\mu - m)\psi'(x') = 0 \tag{10.11}$$

In order to reproduce the Klein-Gordon condition the matrices $\gamma^{\mu\prime}$ must satisfy the same algebra as the matrices γ^μ. By requiring also the same hermiticity condition (10.9), neglecting a unitary transformation, we can always identify $\gamma^{\mu\prime} \equiv \gamma^\mu$.

Assuming that

$$\psi'(x') = S(\Lambda)\psi(x) = S(\Lambda)\psi(\Lambda^{-1}x') \tag{10.12}$$

we can prove that, if for an infinitesimal transformation we write Λ as

$$\Lambda^\mu_{\ \nu} \sim g^\mu_{\ \nu} + \epsilon^\mu_{\ \nu} \tag{10.13}$$

then

$$S(\Lambda) \sim 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} \tag{10.14}$$

with

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \qquad (10.15)$$

with $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$.

In fact from the Dirac equation in the S system, we get

$$\begin{aligned} 0 &= (i\gamma^\mu \partial_\mu - m))\, S^{-1}(\Lambda)\psi'(x') \\ &= \left(i\gamma^\mu \frac{\partial x'^\nu}{\partial x^\mu}\partial'_\nu - m)\right) S^{-1}(\Lambda)\psi'(x') \\ &= \left(i\gamma^\mu \Lambda^\nu{}_\mu \partial'_\nu - m)\right) S^{-1}(\Lambda)\psi'(x') \end{aligned} \qquad (10.16)$$

where $\partial'_\mu = \frac{\partial}{\partial x'^\mu}$. By multiplying by $S(\Lambda)$ to the left, we get

$$\left(iS(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \partial'_\nu - m)\right) \psi'(x') = 0 \qquad (10.17)$$

Comparing with the Dirac equation in the S' system we get

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu = \gamma^\nu \qquad (10.18)$$

or

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu \gamma^\mu \qquad (10.19)$$

For the infinitesimal transformation (10.13) and assuming (10.14) to first order in $\epsilon^{\rho\lambda}$ we get

$$[\sigma_{\rho\lambda}, \gamma_\nu] = -2i(g_{\rho\nu}\gamma_\lambda - g_{\lambda\nu}\gamma_\rho) \qquad (10.20)$$

The unique solution to eq.(10.20) is given by

$$\sigma_{\rho\lambda} = \frac{i}{2}[\gamma_\rho, \gamma_\lambda] \qquad (10.21)$$

This can be verified by noticing that

$$\begin{aligned} \gamma_\rho\gamma_\lambda &= \frac{1}{2}[\gamma_\rho, \gamma_\lambda]_+ + \frac{1}{2}[\gamma_\rho, \gamma_\lambda] \\ &= \frac{1}{2}[\gamma_\rho, \gamma_\lambda]_- + g_{\rho\lambda} \\ &= \frac{1}{i}\sigma_{\rho\lambda} + g_{\rho\lambda} \end{aligned} \qquad (10.22)$$

Then

$$\begin{aligned}[\sigma_{\rho\lambda}, \gamma_\nu] &= i[\gamma_\rho\gamma_\lambda, \gamma_\nu] = i\gamma_\rho[\gamma_\lambda, \gamma_\nu]_+ - i[\gamma_\rho, \gamma_\nu]_+\gamma_\lambda \\ &= -2i(g_{\rho\nu}\gamma_\lambda - g_{\lambda\nu}\gamma_\rho)\end{aligned} \tag{10.23}$$

For a finite transformation the solution for $S(\Lambda)$ is obtained by exponentiation of eq.(10.14):

$$S(\Lambda) = \exp(-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}) \tag{10.24}$$

$S(\Lambda)$ is the representation of the Lorentz transformation Λ on the space of the wave functions ψ. The six matrices $\sigma_{\mu\nu}$ are the generators of the Lorentz transformations. In particular σ_{0i} are the three generators of the Lorentz boosts and σ_{ij} are the three generators of the 3D-rotations. Using the Dirac Pauli representation for the gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{10.25}$$

we have

$$\sigma_{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \tag{10.26}$$

Let us now prove that $\bar{\psi}\psi \equiv \psi^\dagger\gamma^0\psi$ is a scalar under Lorentz transformation. In fact from eq.(10.12)

$$\psi^\dagger \to \psi^\dagger S(\Lambda)^\dagger \tag{10.27}$$

and

$$\bar{\psi} \to \psi^\dagger S(\Lambda)^\dagger\gamma^0 = \bar{\psi}\gamma^0 S(\Lambda)^\dagger\gamma^0 = \bar{\psi}S^{-1}(\Lambda) \tag{10.28}$$

where in the last term we have used the property (10.9). Therefore

$$\bar{\psi}\psi \to \bar{\psi}S^{-1}(\Lambda)S(\Lambda)\psi = \bar{\psi}\psi \tag{10.29}$$

In analogous way we can prove that $\bar{\psi}\gamma^\mu\psi$ transforms as a fourvector. In fact

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi \tag{10.30}$$

where use has been made of (10.19).

Finally we can also prove that $\bar{\psi}\sigma^{\mu\nu}\psi$ transform as a tensor:

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^{\mu}_{\ \rho}\Lambda^{\mu}_{\ \lambda}\bar{\psi}\sigma^{\rho\lambda}\psi \tag{10.31}$$

Example Let us consider a generic z-axis rotation. The $4\times$ 4 matrix representing the z-axis rotation of an angle θ is given by

$$R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.32}$$

For infinitesimal θ we get

$$R_z \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4 + \epsilon \tag{10.33}$$

or

$$\epsilon^{\mu}_{\ \nu} = \theta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.34}$$

From eq.(10.24) we obtain

$$S(R_z) = \exp(-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}) = \exp(-\frac{i}{2}\sigma_{12}\epsilon^{12}) = \exp(\frac{i}{2}\theta\sigma_{12}) \tag{10.35}$$

with

$$\sigma_{12} = \frac{i}{2}[\gamma_1, \gamma_2] = i\gamma_1\gamma_2 = i\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \tag{10.36}$$

In conclusion under a z-rotation of an angle θ the spinor ψ transforms as

$$\psi \to \psi' = \exp\left[\frac{i}{2}\theta\begin{pmatrix} \sigma^3/2 & 0 \\ 0 & \sigma^3/2 \end{pmatrix}\right]\psi = \begin{pmatrix} \exp(i\theta\sigma^3/2) & 0 \\ 0 & \exp(i\theta\sigma^3/2) \end{pmatrix}\psi \tag{10.37}$$

and $\sigma_{12}/2$ is the corresponding generator.

Let us now study the parity transformation

$$\mathbf{x} \to -\mathbf{x} \tag{10.38}$$

which is assumed to be a symmetry of the Dirac equation. The parity transformation acts on the fourdimensional space as

$$x'^{\mu} = P^{\mu}_{\nu} x^{\nu}, \ \ P^{\mu}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{10.39}$$

The transformation of the spinor field under the parity transformation is obtained by requiring the invariance of the Dirac equation under parity and it is given by

$$\psi(x) \to \psi'(x') = S(P)\psi(x) \tag{10.40}$$

with $S(P)$ such that

$$S^{-1}(P)\gamma^0 S(P) = \gamma^0, \ \ S^{-1}(P)\gamma^i S(P) = -\gamma^i \tag{10.41}$$

In this way we obtain:

$$\begin{aligned} &(i\gamma^0\partial_0 - i\gamma^i\partial_i - m)\psi(x) = 0 \\ &\to_P (i\gamma^0\partial_0 + i\gamma^i\partial'_i - m)S^{-1}(P)\psi'(x') = 0 \end{aligned} \tag{10.42}$$

By multiplying by $S(P)$ and using (10.41) we obtain

$$(i\gamma^0\partial_0 - i\gamma^i\partial'_i - m)\psi'(x') = 0 \tag{10.43}$$

The solution of eq.(10.41) is given by

$$S(P) = \gamma^0\eta_P \tag{10.44}$$

with $|\eta_P| = 1$.

It is easy to check that under the parity transformation

$$\bar{\psi}\psi \to \bar{\psi}\psi \tag{10.45}$$

Using the matrix γ_5 defined as[13]

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{10.46}$$

with the properties

$$\gamma_5^\dagger = \gamma_5, \;\; \gamma_5^2 = 1, \;\; [\gamma_5, \gamma_\mu]_+ = 0 \tag{10.47}$$

we can consider the expression

$$\bar{\psi}\gamma_5\psi \tag{10.48}$$

Under the Lorentz transformation

$$\bar{\psi}\gamma_5\psi \to \bar{\psi}\gamma_5\psi \tag{10.49}$$

However under the parity transformation

$$\bar{\psi}\gamma_5\psi \to -\bar{\psi}\gamma_5\psi \tag{10.50}$$

In conclusion while $\bar{\psi}\psi$ is a scalar, the $\bar{\psi}\gamma_5\psi$ bilinear is a pseudo-scalar. In analogous way one can prove that

$$\bar{\psi}\gamma_5\gamma_\mu\psi \tag{10.51}$$

is an axial-four-vector.

Note that the 16 matrices $(1, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu})$ are a basis for the 4 by 4 hermitian matrices.

10.3 Wave plane solutions of the Dirac equation

Let us now look for solutions of the Dirac equation of the form

$$\begin{aligned} \psi(x) &= e^{-ipx}u(\mathbf{p}), \;\; \text{positive energy} \\ \psi(x) &= e^{ipx}v(\mathbf{p}), \;\; \text{negative energy} \end{aligned} \tag{10.52}$$

[13] An alternative form for γ_5 is $\gamma_5 = \frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, with $\epsilon_{0123} = +1$. This form is useful to show that $\bar{\psi}\gamma_5\psi$ trasforms as a pseudoscalar, using $S^{-1}(\Lambda)\gamma_5 S(\Lambda) = (\det\Lambda)\gamma_5 = \gamma_5$, under proper Lorentz transformation and $S^{-1}(P)\gamma_5 S(P) = \gamma_0\gamma_5\gamma_0 = -\gamma_5$ under parity. In the Dirac-Pauli representation $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

where

$$px \equiv p^0 t - \mathbf{p}\cdot\mathbf{x} \equiv Et - \mathbf{p}\cdot\mathbf{x} \tag{10.53}$$

with

$$E = \sqrt{\mathbf{p}^2 + m^2} \tag{10.54}$$

By substituting eqs.(10.52) in the Dirac equation, we get:

$$(\hat{p} - m)u(\mathbf{p}) = 0, \ \ (\hat{p} + m)v(\mathbf{p}) = 0 \tag{10.55}$$

Going into the rest frame $p^\mu = (m, \mathbf{0})$, eqs.(10.55) become

$$(\gamma^0 - 1)u(\mathbf{0}) = 0, \ \ (\gamma^0 + 1)v(\mathbf{0}) = 0 \tag{10.56}$$

In the Dirac-Pauli representation, where γ^0 is given by (10.25) the solutions are

$$u_1(\mathbf{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \ u_2(\mathbf{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \ v_1(\mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \ v_2(\mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{10.57}$$

Note that we have two positive energy and two negative energy independent solutions. $u_{1,2}$ and $v_{1,2}$ are eigenvectors of the generator of rotations along the z axis (see eqs. (10.35) and (10.37),

$$\frac{\sigma_{12}}{2} = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{10.58}$$

corresponding to eigenvalues $\pm 1/2$. Therefore we expect that these solutions represent spin 1/2 particles.

The solutions to a generic frame can be obtained by boosting these solutions from the rest frame with a Lorentz transformation with velocity $\mathbf{v} = \mathbf{p}/E$ (see [2]). However it is simpler to observe that

$$\begin{aligned} (\hat{p} - m)(\hat{p} + m) &= \gamma^\mu p_\mu \gamma^\nu p_\nu - m^2 \\ &= \frac{1}{2}[\gamma^\mu, \gamma^\nu]_+ p_\mu p_\nu - m^2 = p^2 - m^2 = 0 \end{aligned} \tag{10.59}$$

where use has been made of (10.8). Therefore we have

$$u_r(\mathbf{p}) \sim (\hat{p} + m)u_r(\mathbf{0}), \ \ v_r(\mathbf{p}) \sim (\hat{p} - m)v_r(\mathbf{0}), \ \ r = 1, 2 \tag{10.60}$$

By requiring the normalization

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = \delta_{rs}, \quad \bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = -\delta_{rs} \tag{10.61}$$

we get (see Appendix G.1)

$$u_r(\mathbf{p}) = \frac{\hat{p} + m}{\sqrt{2m(E+m)}}u_r(\mathbf{0}), \quad v_r(\mathbf{p}) = \frac{-\hat{p} + m}{\sqrt{2m(E+m)}}v_r(\mathbf{0}), \quad r = 1, 2 \tag{10.62}$$

We have also the orthogonality condition

$$\bar{u}_r(\mathbf{p})v_s(\mathbf{p}) = 0, \quad \bar{v}_r(\mathbf{p})u_s(\mathbf{p}) = 0 \tag{10.63}$$

and the completeness condition

$$\sum_r [u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}) - v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p})] = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4 \tag{10.64}$$

Now the complete set

$$\frac{1}{\sqrt{V}}e^{-ipx}u_r(\mathbf{p}), \quad \frac{1}{\sqrt{V}}e^{ipx}v_r(\mathbf{p}) \tag{10.65}$$

can be used to expand the general solution of the Dirac equation

$$\psi(x) = \sum_{r\mathbf{p}} \sqrt{\frac{m}{VE}}[b_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^*(\mathbf{p})v_r(\mathbf{p})e^{ipx}] \tag{10.66}$$

with $b_r(\mathbf{p}), d_r^*(\mathbf{p})$ complex functions that after quantization of the spinor field ψ become operators. Notice that we have also (see Appendix G.1)

$$u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) = \frac{E}{m}\delta_{rs} = v_r^\dagger(\mathbf{p})v_s(\mathbf{p}) \tag{10.67}$$

$$u^\dagger(\mathbf{p})v(-\mathbf{p}) = v^\dagger(\mathbf{p})u(-\mathbf{p}) = 0 \tag{10.68}$$

10.4 Lagrangian of the Dirac field

The Dirac equation given in eq.(10.6) can be derived from the Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)[i\hat{\partial} - m]\psi(x) \tag{10.69}$$

Varying the action with respect to $\bar{\psi}$, we obtain the Dirac equation. Assuming the mass as fundamental dimension, the dimension of the Dirac field are $M^{3/2}$ so that the action is dimensionless. The interaction of the relativistic electron with the electromagnetic field is obtained by using the *minimal substitution*

$$E \to E - eA^0, \;\; \mathbf{p} \to \mathbf{p} - e\mathbf{A} \tag{10.70}$$

or in the covariant form

$$p^\mu \to p^\mu - eA^\mu \tag{10.71}$$

and

$$i\partial^\mu \to i\partial^\mu - eA^\mu \tag{10.72}$$

where A^μ is the four potential of the electromagnetic field. The Lagrangian describing the Dirac field interacting with the electromagnetic field is therefore

$$\mathcal{L}' = \bar{\psi}(x)[i\hat{\partial} - e\hat{A} - m]\psi(x) \tag{10.73}$$

and the corresponding Dirac equation is

$$[i\hat{\partial} - e\hat{A} - m]\psi(x) = 0 \tag{10.74}$$

10.5 Non relativistic limit of the Dirac equation

Let us now consider the non relativistic limit of the Dirac equation in an external field. Starting from (10.74) and multiplying by γ_0 we get

$$i\frac{\partial\psi(x)}{\partial t} = [\boldsymbol{\alpha} \cdot (-i\boldsymbol{\nabla} - e\mathbf{A}) + \beta m + eA_0]\psi(x) \tag{10.75}$$

Let us now look for solutions with positive energy of the following form

$$\psi(x) = \exp(-iEt)\begin{pmatrix} \phi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{pmatrix} \tag{10.76}$$

where ϕ, χ are two component spinors depending on $\boldsymbol{x}$ and

$$E = m + T \tag{10.77}$$

We obtain

$$(m+T)\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}(m+eA_0)\phi + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi\\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\phi + (-m+eA_0)\chi\end{pmatrix} \tag{10.78}$$

with $\boldsymbol{\pi} = -i\boldsymbol{\nabla} - e\mathbf{A}$ ($\pi^i = -i\partial_i - eA^i$) or

$$\begin{aligned}(eA_0 - T)\phi + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\chi &= 0\\ \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\phi + (-2m + eA_0 - T)\chi &= 0\end{aligned} \tag{10.79}$$

By considering the non relativistic limit, $A_0, T << m$, from the second equation of (10.79) we get

$$\chi = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\phi}{2m} \tag{10.80}$$

By substituting (10.80) in the first equation of (10.79) we obtain

$$T\phi = (\frac{1}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + eA_0)\phi \tag{10.81}$$

Using

$$\begin{aligned}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\pi} &= \sigma^i\sigma^j\pi^i\pi^j = \frac{1}{2}([\sigma^i,\sigma^j] + [\sigma^i,\sigma^j]_+)\pi^i\pi^j = i\epsilon_{ijk}\sigma^k\pi^i\pi^j + \boldsymbol{\pi}^2\\ &= \frac{i}{2}\epsilon_{ijk}\sigma^k[\pi^i,\pi^j] + \boldsymbol{\pi}^2\\ &= -\frac{1}{2}e\epsilon_{ijk}\sigma^k(\partial_i A^j - \partial_j A^i) + \boldsymbol{\pi}^2\\ &= -e\epsilon_{ijk}\sigma^k\partial_i A^j + \boldsymbol{\pi}^2\\ &= -e\boldsymbol{\sigma}\cdot\boldsymbol{B} + \boldsymbol{\pi}^2\end{aligned} \tag{10.82}$$

where we have used $[-i\partial_i, f(x)] = [-i\partial_i, x^l]\partial f/\partial x^l = -i\partial f/\partial x^i$. We get

$$\begin{aligned}T\phi = \quad &(\frac{1}{2m}\boldsymbol{\pi}^2 - \frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B} + eA_0)\phi\\ = \quad &(\frac{1}{2m}\boldsymbol{\pi}^2 - \frac{e}{2m}2\boldsymbol{S}\cdot\boldsymbol{B} + eA_0)\phi\\ = \quad &(\frac{1}{2m}\boldsymbol{\pi}^2 - \mu\cdot\boldsymbol{B} + eA_0)\phi\end{aligned} \tag{10.83}$$

where $\boldsymbol{S}$ denotes the electron spin and $\boldsymbol{\mu}$ the magnetic momentum of the electron

$$\boldsymbol{\mu} = \frac{e}{2m} 2\boldsymbol{S} \tag{10.84}$$

In conclusion Dirac equation predicts the magnetic field the gyromagnetic factor of the electron

$$g_e = 2 \tag{10.85}$$

Its experimental value is different from 2 at the per mil level [18]

$$\frac{g_e - 2}{2} = 1159.65218076 \pm 0.00000027 \times 10^{-6} \tag{10.86}$$

By studying the next order in the non relativistic expansion of the Dirac equation and assuming for A^0 the Coulomb potential for the Hydrogen atom, one can get the fine structure terms, see [17]: the relativistic correction

$$-\frac{p^4}{8m^3} \tag{10.87}$$

the Darwin term

$$\frac{e}{8m^2} \nabla^2 A_0 \tag{10.88}$$

and the spin-orbit term

$$\frac{e}{2m^2} \frac{1}{r} \frac{d\phi}{dr} \boldsymbol{S} \cdot \boldsymbol{L} \tag{10.89}$$

The exact solution of the Dirac equation and the relativistic form of the energy levels for the Hydrogen atom can be found for instance in [2].

10.6 Quantization of the Dirac field

The Dirac equation given in eq.(10.6) can be derived from the Lagrangian density (10.69) by minimizing the action with respect to $\bar{\psi}$. The conjugate momenta are given by

$$\pi_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = i\psi_\alpha^\dagger, \quad \bar{\pi}_\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}_\alpha} = 0 \tag{10.90}$$

The Hamiltonian is given by

$$H = \int d^3x \mathcal{H} \tag{10.91}$$

with

$$\mathcal{H} = \pi(x)\dot{\psi}(x) + \bar{\pi}(x)\dot{\bar{\psi}}(x) - \mathcal{L} = \bar{\psi}(x)[-i\gamma^j\partial_j + m]\psi(x) \tag{10.92}$$

When computing the Hamiltonian, using the field expansion (10.66), we have $\mathcal{L} = 0$ and therefore the Hamiltonian is simply given by

$$H = i\int d^3x \psi^\dagger(x)\dot{\psi}(x) \tag{10.93}$$

Therefore substituting in the Hamiltonian the expansion (10.66) we obtain

$$\begin{aligned} H &= \int d^3x \sum_{r\mathbf{p}} \sqrt{\frac{m}{VE}}[b_r^\dagger(\mathbf{p})u_r^\dagger(\mathbf{p})e^{ipx} + d_r(\mathbf{p})v_r^\dagger(\mathbf{p})e^{-ipx}] \\ & \sum_{s\mathbf{p}'} E'\sqrt{\frac{m}{VE'}}[b_s(\mathbf{p}')u_s(\mathbf{p}')e^{-ip'x} - d_s^\dagger(\mathbf{p}')v_s(\mathbf{p}')e^{ip'x}] \\ &= m\sum_{r\mathbf{p}}\sum_{s\mathbf{p}'}\frac{E'}{\sqrt{EE'}}\Big[\Big(b_r^\dagger(\mathbf{p})b_s(\mathbf{p}')u_r^\dagger(\mathbf{p})u_s(\mathbf{p}') \\ &- d_r(\mathbf{p})d_s^\dagger(\mathbf{p}')v_r^\dagger(\mathbf{p})v_s(\mathbf{p}')\Big)\delta_{\mathbf{p},\mathbf{p}'} \\ &+ \Big(d_r(\mathbf{p})b_s(\mathbf{p}')v_r^\dagger(\mathbf{p})u_s(\mathbf{p}')e^{-2iEt} - b_r^\dagger(\mathbf{p})d_s^\dagger(\mathbf{p}')u_r^\dagger(\mathbf{p})v_s(\mathbf{p}')e^{2iEt}\Big)\delta_{\mathbf{p},-\mathbf{p}'}\Big] \end{aligned} \tag{10.94}$$

and using the orthogonality properties of the spinors, eq.(10.67) and eq.(10.68),

$$H = \sum_{\mathbf{p}r} E[b_r^\dagger(\mathbf{p})b_r(\mathbf{p}) - d_r(\mathbf{p})d_r^\dagger(\mathbf{p})] \tag{10.95}$$

If we now would assume commutation relations

$$[b_r(\mathbf{p}), b_s^\dagger(\mathbf{p}')] = [d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')] = \delta_{rs}\delta_{\mathbf{p},\mathbf{p}'} \tag{10.96}$$

the Hamiltonian could be rewritten, apart an infinite term, as

$$H = \sum_{r\mathbf{p}} E[b_r^\dagger(\mathbf{p})b_r(\mathbf{p}) - d_r^\dagger(\mathbf{p})d_r(\mathbf{p})] \tag{10.97}$$

This Hamiltonian is unbounded from below and therefore the theory does not admit a stable minimum. However, if we assume anticommutation relations

$$[b_r(\mathbf{p}), b_s^\dagger(\mathbf{p}')]_+ = [d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')]_+ = \delta_{rs}\delta_{\mathbf{p},\mathbf{p}'} \tag{10.98}$$

and all other anticommutators vanishing, the expression for H is, apart an infinite term,

$$H = \sum_{r\mathbf{p}} E[b_r^\dagger(\mathbf{p})b_r(\mathbf{p}) + d_r^\dagger(\mathbf{p})d_r(\mathbf{p})] \tag{10.99}$$

which is now definite positive and admits a minimum state $|0>$ with zero energy. The state $|0>$ is defined by the conditions

$$b_r(\mathbf{p})|0>= d_r(\mathbf{p})|0>= 0 \tag{10.100}$$

Dirac quantized theory describes two types of particles: one can build one particle states as

$$b_r^\dagger(\mathbf{p})|0>, \text{ and } d_r^\dagger(\mathbf{p})|0> \tag{10.101}$$

To distinguish these two types of particles we can consider additional operators commuting with H. Since the theory is invariant under gauge transformations

$$\psi \to e^{i\alpha}\psi, \;\; \psi^\dagger \to e^{-i\alpha}\psi^\dagger \tag{10.102}$$

we can build the corresponding Noether current (see eq.(3.80))

$$j^\mu = \frac{\partial \mathcal{L}}{\partial\partial_\mu\psi}\Delta\psi + \Delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}} \tag{10.103}$$

which turns out to be

$$j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x) \tag{10.104}$$

This current can also be identified by introducing in the Dirac Lagrangian the interaction with the electromagnetic field by means of the minimal subsitution $i\partial_\mu \to i\partial_\mu - eA_\mu$ as done in Section 10.4.

Let us compute the total charge as

$$Q = \int d^3x j^0 = \int d^3x \psi^\dagger(x)\psi(x) \tag{10.105}$$

By substituting the expansion (10.66) in Q we obtain, again neglecting an infinite term,

$$Q = \sum_{r\mathbf{p}} [b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) - d_r^\dagger(\mathbf{p}) d_r(\mathbf{p})] \tag{10.106}$$

Therefore the two types of particles have opposite charges. In conclusion the Dirac equation describes the electron and its antiparticle the positron. This particle was discovered in 1932 by the american physicist Carl Anderson who received the Nobel prize in Physics in 1936. This discovery had been made earlier by P. Blackett[14] and G. Occhialini[15] who however did not immediately publish their results. Dirac was awarded of the Nobel in 1933 together with Schrödinger *"for the discovery of new productive forms of atomic theory"*.

Quantization of a field theory with anticommutators implies Fermi Pauli statistics: the two particle state is antisymmetric under the exchange of the two particles

$$b^\dagger_{\mathbf{p}1,r} b^\dagger_{\mathbf{p}2,s}|0> = -b^\dagger_{\mathbf{p}2,s} b^\dagger_{\mathbf{p}1,r}|0> \tag{10.107}$$

Furthermore since $[b^\dagger_{\mathbf{p},r}]^2 = 0$ it is impossible to build a state with two electrons with the same quantum numbers.

Using the invariance of the action under translation and Lorentz transformations we can also consider the total spatial momentum

$$\mathbf{P} = -i \int d^3x \psi^\dagger \boldsymbol{\nabla} \psi = \sum_{r\mathbf{p}} \mathbf{p}[b_r^\dagger(\mathbf{p}) b_r(\mathbf{p}) + d_r^\dagger(\mathbf{p}) d_r(\mathbf{p})] \tag{10.108}$$

and the angular momentum:

$$\mathbf{M} = \int d^3x \psi^\dagger(x)[\mathbf{x} \times (-i\nabla)]\psi(x) + \int d^3x \psi^\dagger(x) \frac{1}{2} \boldsymbol{\sigma} \psi(x) \tag{10.109}$$

where $\boldsymbol{\sigma}$ a 4 by 4 matrix given by

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{10.110}$$

The two terms represent the angular momentum and the spin term for 1/2 particles.

[14] Patrick Blackett, 1897-1974
[15] Giuseppe Occhialini,1907-1993

Finally let us note that using the anticommutation relations (10.98) and the expansion (10.66), one can derive the canonical anticommutation relations

$$[\psi(t,\mathbf{x}),\Pi(t,\mathbf{y})]_+ = i\delta^3(x-y), \quad [\psi(t,\mathbf{x}),\psi(t,\mathbf{y})]_+ = [\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})]_+ = 0 \tag{10.111}$$

10.7 Coulomb scattering of electrons

As an application of Quantum Field Theory, let us now consider the scattering of a relativistic electron by a classical Coulomb potential generated by a point charge $-Ze > 0$. The Hamiltonian interaction density is given by, using eq.(10.73),

$$\mathcal{H}_I = -\mathcal{L}_I = e\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x) \tag{10.112}$$

with

$$A^\mu(x) = (-\frac{Ze}{4\pi|\mathbf{x}|}, \mathbf{0}) \tag{10.113}$$

Let us consider, as initial state, an electron with four-momentum p and polarization r

$$|e(p,r)>= b_r^\dagger(\mathbf{p})|0> \tag{10.114}$$

and, as a final state, an electron with four-momentum p' and polarization s

$$|e(p',s)>= b_s^\dagger(\mathbf{p}')|0> \tag{10.115}$$

and consider the matrix element

$$\begin{aligned} V_{fi} &= < e(p',s)|H_I|e(p,r) > \\ &= \int d^3x < e(p',s)|e\bar{\psi}(x)\gamma^0\psi(x)A^0(x)|_{t=0}|e(p,r) > \\ &= e\sqrt{\frac{m_e}{E_pV}}\sqrt{\frac{m_e}{E_{p'}V}}\int d^3x e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}\bar{u}_s(\mathbf{p}')\gamma^0u_r(\mathbf{p})A^0(\mathbf{x}) \end{aligned} \tag{10.116}$$

where we have used the spinor expansion (10.66) and the anticommutation relations (10.98).

By introducing the transfer momentum

$$\mathbf{q} = \mathbf{p} - \mathbf{p}' \tag{10.117}$$

we can rewrite V_{fi} as

$$V_{fi} \quad = \quad e\sqrt{\frac{m}{E_p V}}\sqrt{\frac{m}{E_{p'} V}}\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p})A^0(\mathbf{q}) \tag{10.118}$$

where we have introduced the Fourier transform of the Coulomb potential $A^0(\mathbf{x})$

$$A^0(\mathbf{q}) = \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} A^0(\mathbf{x}) \tag{10.119}$$

Its explicit form is

$$A^0(\mathbf{q}) = -\frac{Ze}{\mathbf{q}^2} \tag{10.120}$$

Using the Fermi rule the cross section is then given by

$$d\sigma = 2\pi\delta(E_{p'} - E_p)|V_{fi}|^2 \frac{V d^3p'}{(2\pi)^3}\frac{1}{\Phi} \tag{10.121}$$

where

$$\Phi = \frac{|\mathbf{p}|}{E_p V} \tag{10.122}$$

is the incident flux. We obtain

$$d\sigma = \delta(E_{p'} - E_p)\frac{m^2 Z^2 e^4}{(2\pi)^2} d^3p' \frac{|\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p})|^2}{|\mathbf{p}| E_p \mathbf{q}^4} \tag{10.123}$$

or

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \delta(E_{p'} - E_p)\frac{m^2 Z^2 e^4}{(2\pi)^2} p'^2 dp' \frac{|\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p})|^2}{|\mathbf{p}| E_p \mathbf{q}^4} \\ &= \delta(E_{p'} - E_p) 4m^2 Z^2 \alpha^2 \frac{|\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p})|^2}{\mathbf{q}^4} dE_{p'} \end{aligned} \tag{10.124}$$

where we have used

$$|\mathbf{p}'| = |\mathbf{p}|, \;\; E_p = E_{p'} \tag{10.125}$$

and

$$\alpha = \frac{e^2}{4\pi} \tag{10.126}$$

By integrating over $dE_{p'}$ we get

$$\frac{d\sigma}{d\Omega} = 4m^2 Z^2 \alpha^2 \frac{|\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p})|^2}{\mathbf{q}^4} \tag{10.127}$$

If we do not know the initial polarization and we do not measure the final electron polarization, we average over the initial spin, assuming equal a priori probability to different initial polarization states, and sum over the final spin

$$\frac{d\sigma_{unpol}}{d\Omega} = 4m^2Z^2\alpha^2\frac{1}{2}\sum_{r,s}\frac{|\bar{u}_s(\mathbf{p}')\gamma^0 u(\mathbf{p})|^2}{\mathbf{q}^4} \tag{10.128}$$

The polarization sums can be reduced to traces as we are going to see. Let us consider

$$\begin{aligned}
\sum_{r,s}|\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p})|^2 &= \sum_{r,s}\bar{u}_r(\mathbf{p})\gamma^0 u_s(\mathbf{p}')\bar{u}_s(\mathbf{p}')\gamma^0 u_r(\mathbf{p}) \\
&= \sum_{r,s,\alpha,\beta,\gamma,\delta}\bar{u}_{r\alpha}(\mathbf{p})\gamma^0_{\alpha\beta}u_{s\beta}(\mathbf{p}')\bar{u}_{s\gamma}(\mathbf{p}')\gamma^0_{\gamma\delta}u_{r\delta}(\mathbf{p}) \\
&= \gamma^0_{\alpha\beta}(\frac{\hat{p}'+m}{2m})_{\beta\gamma}\gamma^0_{\gamma\delta}(\frac{\hat{p}+m}{2m})_{\delta\alpha} \\
&= Tr(\gamma^0\frac{\hat{p}'+m}{2m}\gamma^0\frac{\hat{p}+m}{2m}) \\
&= \frac{-p\cdot p'+2p_0p_0'+m^2}{m^2}
\end{aligned} \tag{10.129}$$

where use have been made of the definition of positive energy projectors (G.22) and of (see (G.34))

$$Tr[\gamma^0\gamma^\mu\gamma^0\gamma^\nu] = -4g^{\mu\nu} + 8g^{0\mu}g^{0\nu} \tag{10.130}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho] = 0 \tag{10.131}$$

The numerator of Eq. (10.129) can be written as

$$\begin{aligned}
-p\cdot p'+2p_0p_0'+m^2 &= -E_pE_p'+\mathbf{p}'\cdot\mathbf{p}+2E_pE_p'+m^2 \\
&= E_pE_p'+\mathbf{p}'\cdot\mathbf{p}+m^2 \\
&= E_p^2+\mathbf{p}^2\cos\theta+m^2
\end{aligned} \tag{10.132}$$

where θ is the scattering angle. By substituting in eq.(10.128), we obtain the final result for the differential unpolarized scattering cross section

$$\begin{aligned}
\frac{d\sigma}{d\Omega}_{unpol} &= \frac{\alpha^2Z^2}{8\mathbf{p}^4\sin^4\frac{\theta}{2}}(E_p^2+\mathbf{p}^2\cos\theta+m^2) \\
&= \frac{\alpha^2Z^2}{4E_p^2\mathbf{v}^4\sin^4\frac{\theta}{2}}(1-\mathbf{v}^2\sin^2\frac{\theta}{2})
\end{aligned} \tag{10.133}$$

where we have used

$$\mathbf{q}^2 = 4\mathbf{p}^2 \sin^2 \frac{\theta}{2} \qquad (10.134)$$

The cross section given by eq.(10.133) is the Mott cross section. For $\mathbf{v} << 1$ the formula reduces to the Rutherford scattering.

10.8 Higgs decay width to fermions

As a second application of the perturbative approach of Quantum Field Theory we consider the decay rate of the Higgs[16] in electron and positron, $H \to e^+e^-$. The Higgs is a spin zero particle, present in the spectrum of the Standard Model (SM) of electroweak interactions and responsible for the mechanism of generating the masses of the quarks u, d, c, s, t, b, of the leptons $e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$ and of the gauge fields $W^\pm, Z$. The experimentally measured Higgs mass is $m_H \sim 125.1$ GeV [18].

Quarks and leptons have spin 1/2 and therefore can be described by Dirac fields. The Higgs is not a stable particle and decays in various channels. His main decay is in the quark channel $\bar{b}b$ where $\bar{b}$ denotes the antiparticle of the quark b. The branching ratio in the $\bar{b}b$ channel is defined as

$$B(H \to \bar{b}b) = \frac{\Gamma(H \to \bar{b}b)}{\Gamma_H^{tot}} \qquad (10.135)$$

where Γ_H^{tot} is the total width of the Higgs, which, within the SM, is predicted to be $\Gamma_H^{tot} = 4.2$ MeV. The branching ratio in the $\bar{b}b$ channel is approximately 58 %[18], corresponding to $\Gamma(H \to \bar{b}b) = 2.44$ MeV.

However this channel has a large background from strong interactions. The relevant decay channels for Higgs discovery are

$$\gamma\gamma, W^+W^-, ZZ, \tau^+\tau^-, \bar{b}b \qquad (10.136)$$

In the following we compute the $H \to e^+e^-$ width and then with a suitable rescaling the $H \to \bar{b}b$ width.

[16] P. Higgs (1929-), Nobel prize in Physics in 2013

The Higgs interaction with the spin 1/2 particles (quarks and leptons) is decribed by an Hamiltonian (see for example [14])

$$H_I = \int d^3x \mathcal{H_I} \tag{10.137}$$

with

$$\mathcal{H_I} = -\lambda\phi(x)\bar{\psi}(x)\psi(x) \tag{10.138}$$

where $\phi(x)$ is the scalar field which describes the Higgs and $\psi(x)$ the Dirac field describing the fermion. The interaction coupling is

$$\lambda = \frac{m}{v} \sim \frac{m}{246 \text{ GeV}} \tag{10.139}$$

where m is the mass of the field ψ. The coupling constant λ is dimensionless. In fact, assuming the mass as fundamental dimension, the dimensions of the scalar field are M^1, the dimensions of the fermion field $M^{3/2}$ and since

$$\mathcal{H}_I = -\mathcal{L}_I \tag{10.140}$$

the dimensions of the Lagrangian are M^4 so that the action is dimensionless. In eq. (10.139) the parameter $v \sim 246$ GeV is related to the energy scale (the inverse of the range) of the weak interactions, $G_F^{-1/2}$. The Fermi constant G_F is related to v by

$$G_F = \frac{1}{\sqrt{2}v^2} \tag{10.141}$$

The masses of the three gauge bosons $W^\pm, Z$ which mediate the weak interactions are of the order of v. In particular $m_W = 80.379 \pm 0.012$ GeV, $m_Z = 91.1876 \pm 0.0021$ GeV [18].

For the electron, $m_e \sim 0.5$ MeV, $\lambda \sim 10^{-6}$, therefore the decay $H \to e^+e^-$ is very small. However in the case of the μ, $m_\mu = 105.6$ MeV $\sim$200 m_e and the decay rate is larger; LHC has already some evidence for the decay $H \to \mu^+\mu^-$ [19]. For the calculation of the decay widths we will consider the fields quantized in a box of volume V. Therefore we have the following expansions

$$\begin{aligned}
\phi(x) &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2E_kV}}(a_k e^{-ikx} + h.c.) \\
\psi(x) &= \sum_{\mathbf{k}r} \sqrt{\frac{m}{E_kV}}[b_r(\mathbf{k})u_r(\mathbf{k})e^{-ikx} + d_r^\dagger(\mathbf{k})v_r(\mathbf{k})e^{ikx}]
\end{aligned} \tag{10.142}$$

where we recall that

$$kx \equiv E_k t - \mathbf{k} \cdot \mathbf{x} \tag{10.143}$$

and $E_k = \sqrt{\mathbf{k}^2 + m^2}$ where m denotes the corresponding mass of the boson or of the fermion.

The initial state is therefore a Higgs with momentum $\mathbf{p}$

$$|i\rangle = a^\dagger_{\mathbf{p}}|0\rangle \tag{10.144}$$

while the final state is a positron with momentum and $\mathbf{k}_1$ and an electron with momentum $\mathbf{k}_2$

$$|f\rangle = d^\dagger_{r_1}(\mathbf{k}_1) b^\dagger_{r_2}(\mathbf{k}_2)|0\rangle \tag{10.145}$$

where r_1, r_2 are the spin labels. The vacuum is the direct product of the vacua of the two Fock spaces

$$|0\rangle \equiv |0\rangle_\phi \otimes |0\rangle_\psi \tag{10.146}$$

We have, evaluating the Hamiltonian density at $t = 0$,

$$\langle f|\mathcal{H}_I|i\rangle = -\lambda \sqrt{\frac{m_e}{E_{k_1}V}} \sqrt{\frac{m_e}{E_{k_2}V}} \frac{1}{\sqrt{2E_p V}} e^{i(\mathbf{p}-\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{x}} \bar{u}_{r_2}(\mathbf{k}_2) v_{r_1}(\mathbf{k}_1) \tag{10.147}$$

and

$$V_{fi} = \int d^3x \langle f|\mathcal{H}_I|i\rangle = -\lambda \sqrt{\frac{m_e}{E_{k_1}V}} \sqrt{\frac{m_e}{E_{k_2}V}} \frac{1}{\sqrt{2E_p V}} \bar{u}_{r_2}(\mathbf{k}_2) v_{r_1}(\mathbf{k}_1) V \delta_{\mathbf{p},\mathbf{k}_1+\mathbf{k}_2} \tag{10.148}$$

Using the golden rule, we can compute the probability that the Higgs of momentum p decays in a positron of momentum k_1 and an electron of momentum k_2

$$\begin{aligned} dw_{fi} &= 2\pi\delta(E_{k_1} + E_{k_2} - E_p)|V_{fi}|^2 V \frac{d^3k_1}{(2\pi)^3} V \frac{d^3k_2}{(2\pi)^3} \\ &= (2\pi)^4 \delta^4(P_f - P_i) V \lambda^2 \left(\sqrt{\frac{m_e}{E_{k_1}V}} \sqrt{\frac{m_e}{E_{k_2}V}} \frac{1}{\sqrt{2E_p V}}\right)^2 \\ &\quad |\bar{u}_{r_2}(\mathbf{k}_2) v_{r_1}(\mathbf{k}_1)|^2 V \frac{d^3k_1}{(2\pi)^3} V \frac{d^3k_2}{(2\pi)^3} \end{aligned} \tag{10.149}$$

where passing to the continuum we have used

$$(V\delta_{\mathbf{p},\mathbf{k}_1+\mathbf{k}_2})^2 \to (2\pi)^3\delta^3(p-k_1-k_2)V \tag{10.150}$$

$P_{f,i}$ are the final and initial total four-momentum. In order to compute the total width $\Gamma(H \to e^+e^-)$, we can now integrate over $\mathbf{k}_1$ and $\mathbf{k}_2$

$$\begin{aligned}\int dw_{fi} &= \frac{1}{(2\pi)^2}\frac{\lambda^2}{2}\int d^3k_1 d^3k_2 \delta^4(k_1+k_2-p)\frac{m_e^2}{E_{k_1}E_{k_2}E_p} \\ &\quad |\bar{u}_{r_2}(\mathbf{k}_2)v_{r_1}(\mathbf{k}_1)|^2 \\ &= \frac{1}{(2\pi)^2}\frac{\lambda^2}{2}\int d^3k_1 \delta(E_{k_1}+E_{k_2}-E_p)\frac{m_e^2}{E_{k_1}E_{k_2}E_p} \\ &\quad |\bar{u}_{r_2}(\mathbf{k}_2)v_{r_1}(\mathbf{k}_1)|^2 \end{aligned} \tag{10.151}$$

and sum over r_1 and r_2.

$$\begin{aligned}\Gamma(H \to e^+e^-) &= \sum_{r_1r_2}\int_{k_1,k_2} dw_{fi} \\ &= \frac{1}{(2\pi)^2}\frac{\lambda^2}{2}\int d^3k_1 \delta(E_{k_1}+E_{k_2}-E_p)\frac{m_e^2}{E_{k_1}E_{k_2}E_p} \\ &\quad \sum_{r_1r_2}\bar{v}_{r_1}(\mathbf{k}_1)u_{r_2}(\mathbf{k}_2)\bar{u}_{r_2}(\mathbf{k}_2)v_{r_1}(\mathbf{k}_1) \\ &= \frac{1}{(2\pi)^2}\frac{\lambda^2}{2}\int d^3k_1 \delta(E_{k_1}+E_{k_2}-E_p)\frac{m_e^2}{E_{k_1}E_{k_2}E_p} \\ &\quad \frac{1}{4m_e^2}Tr[(\hat{k}_1-m_e)(\hat{k}_2+m_e)] \\ &= \frac{\lambda^2}{8\pi^2}\int d^3k_1 \delta(E_{k_1}+E_{k_2}-E_p)\frac{k_1\cdot k_2 - m_e^2}{E_{k_1}E_{k_2}E_p} \end{aligned} \tag{10.152}$$

In the previous equations we have used

$$\begin{aligned}\sum_{r_1r_2}\bar{v}_{r_1}(\mathbf{k}_1)u_{r_2}(\mathbf{k}_2)\bar{u}_{r_2}(\mathbf{k}_2)v_{r_1}(\mathbf{k}_1) &= \sum_{r_1}v_{r_1\beta}(\mathbf{k}_1)\bar{v}_{r_1\alpha}(\mathbf{k}_1)\sum_{r_2}u_{r_2\alpha}(k_2)\bar{u}_{r_2\beta}(\mathbf{k}_2) \\ &= -Tr(\Lambda_-\Lambda_+) \end{aligned} \tag{10.153}$$

where $\Lambda_\pm = \frac{1}{2m_e}(\pm\hat{k}+m_e)$ are the positive (negative) energy solution projectors. We have also used

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \ \ \mathrm{Tr}(\gamma^\mu) = 0, \ \ \mathrm{Tr}(I) = 4 \tag{10.154}$$

Usually one computes the decay rate in the rest frame of the decaying particle. Therefore

$$p = (m_H, \mathbf{0}), \;\; k_1 = (E_{k_1}, \mathbf{k}_1), \;\; k_2 = (E_{k_2}, -\mathbf{k}_1) \tag{10.155}$$

with

$$E_{k_1} = E_{k_2} = \sqrt{m_e^2 + \mathbf{k}_1^2} \tag{10.156}$$

We have

$$k_1 \cdot k_2 = 2\mathbf{k}_1^2 + m_e^2 \tag{10.157}$$

Substituting in the width, after integration over $d\Omega$ we get

$$\begin{aligned}
\Gamma(H \to e^+e^-) &= \frac{\lambda^2}{m_H \pi}\int_0^\infty dk_1 \frac{k_1^4}{k_1^2 + m_e^2}\delta(m_H - 2\sqrt{k_1^2 + m_e^2}) \\
&= \frac{4\lambda^2}{m_H^3 \pi}\int_0^\infty dk_1 k_1^4 \delta(m_H - 2\sqrt{k_1^2 + m_e^2}) \\
&= \frac{\lambda^2}{m_H^2 \pi}\int_0^\infty dk_1 k_1^3 \delta(k_1 - \frac{1}{2}\sqrt{m_H^2 - 4m_e^2}) \\
&= \frac{\lambda^2}{8\pi} m_H (1 - \frac{4m_e^2}{m_H^2})^{3/2}
\end{aligned} \tag{10.158}$$

where in the last equation we have used the property

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0) \tag{10.159}$$

where x_0 is a zero of $f(x)$ of order one. Using $m_H \sim 125$ GeV and $m_e \sim 0.5$ MeV we get

$$\Gamma(H \to e^+e^-) \sim 2.1 \times 10^{-11} \text{ GeV} = 2.1 \times 10^{-2} \text{ eV} \tag{10.160}$$

For the quark bottom ($m_b = 4.2$ GeV) we obtain, taking into account a factor three from the sum over the three colors of the bottom quark (the color gauge interaction which is responsible of the strong force is based on the $SU(3)$ group and the b quark is a triplet of $SU(3)$)

$$\Gamma(H \to \bar{b}b) \sim 3(\frac{m_b}{m_e})^2 \Gamma(H \to e^+e^-) \sim 4.3 \text{ MeV} \tag{10.161}$$

This result is only approximate because one needs to take into account also QCD and electroweak corrections to $\Gamma(H \to \bar{b}b)$. This channel is the main decay channel of the Higgs however it is not the cleanest because of QCD background. The main discovery channels at LHC were $\gamma\gamma$, ZZ^* and W^+W^{-*} where the $*$ denotes a *virtual* particle [18]. For example, when considering $H \to ZZ$ since $m_H \leq 2m_Z$ only one Z particle can be real. The second Z is only a virtual intermediate state decaying in two particles which are detected in the detectors ATLAS and CMS of LHC.

10.9 The decay width for the process $\mu^- \to e^-\bar{\nu}_e\nu_\mu$

The decay $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ is due to the weak interactions and can be explained by the Fermi interaction[17] between the Dirac fields representing the muon, the electron and the two neutrinos.The effective Hamiltonian density is given by a current-current interaction

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}}(J_e^{\dagger\lambda}J_{\mu\lambda} + h.c) \tag{10.162}$$

where the electronic and the muonic currents are

$$J_e^\lambda = \bar{\psi}_e\gamma^\lambda(1-\gamma_5)\psi_{\nu_e},\ J_\mu^\lambda = \bar{\psi}_\mu\gamma^\lambda(1-\gamma_5)\psi_{\nu_\mu} \tag{10.163}$$

The Fermi constant G_F can be derived, as we will see, by the experimental value of the decay width and turns out to be $G_F = 1.16 \times 10^{-5}$ GeV^{-2}. Remember that each fermion field ψ describes both particle and antiparticles

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x) \tag{10.164}$$

where $\psi^{(\pm)}$ denote the positive and negative energy part of the spinor.

The initial state is a muon with given momentum and spin

$$|\mu^-(p_\mu, r_\mu) >\equiv |\mu^-(p_\mu, r_\mu) > \otimes|0>_{e^-} \otimes|0>_{\nu_\mu} \otimes|0>_{\nu_e} \tag{10.165}$$

and the final state is

$$|e\,\nu_\mu\bar{\nu}_e >= |0>_\mu \otimes|e_-(p_e, r_e) > \otimes|\nu_\mu(p_{\nu_\mu}, r_{\nu_\mu}) > \otimes|\bar{\nu}_e(p_{\bar{\nu}_e}, r_{\bar{\nu}_e}) > \tag{10.166}$$

[17] The theory was proposed by Fermi in 1934

Each fermion field can be expanded as

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_r \sum_{\mathbf{p}} \sqrt{\frac{m}{E}} [b_r(\mathbf{p}) u_r(p) e^{-ipx} + d_r^\dagger(p) v_r(\mathbf{p}) e^{ipx}] \tag{10.167}$$

One can verify that

$$\psi_\mu^{(+)}(x)|\mu_-(p_\mu, r_\mu)> = \frac{1}{\sqrt{V}} \sqrt{\frac{m_\mu}{E_\mu}} u_{r_\mu}(\mathbf{p}_\mu) e^{-ip_\mu x}|0> \tag{10.168}$$

$$< e(p_e, r_e)|(\psi_e^{(+)})^\dagger(x) = \frac{1}{\sqrt{V}} \sqrt{\frac{m_e}{E_e}} u_{r_e}^\dagger(\mathbf{p}_e) e^{ip_e x} < 0| \tag{10.169}$$

and similarly for $\bar{\nu}_e$ and ν_μ.

$$< \bar{\nu}_e(p_{\bar{\nu}_e}, r_{\bar{\nu}_e})|(\psi_{\nu_e}^{(-)})(x) = \frac{1}{\sqrt{V}} \sqrt{\frac{m_{\nu_e}}{E_{\bar{\nu}_e}}} v_{r_{\nu_e}}(\mathbf{p}_{\bar{\nu}_e}) e^{ip_{\bar{\nu}_e} x} < 0| \tag{10.170}$$

$$< \nu_\mu(p_{\nu_\mu}, r_{\nu_\mu})|(\psi_{\nu_\mu}^{(+)})^\dagger(x) = \frac{1}{\sqrt{V}} \sqrt{\frac{m_{\nu_\mu}}{E_{\nu_\mu}}} u_{r_{\nu_\mu}}^\dagger(\mathbf{p}_{\nu_\mu}) e^{ip_{\nu_\mu} x} < 0| \tag{10.171}$$

where

$$|0> \equiv |0>_\mu \otimes |0>_{e^-} \otimes |0>_{\nu_\mu} \otimes |0>_{\nu_e} \tag{10.172}$$

The relevant term of the interaction (10.162) is

$$\begin{aligned} J_{\mu\lambda}^\dagger J_e^\lambda &= \psi_{\nu_\mu}^\dagger (1-\gamma_5)\gamma_\lambda^\dagger \gamma_0 \psi_\mu \psi_e^\dagger \gamma_0 \gamma^\lambda (1-\gamma_5)\psi_{\nu_e} \\ &= \psi_{\nu_\mu}^{(+)\dagger}(1-\gamma_5)\gamma_\lambda^\dagger \gamma_0 \psi_\mu^{(+)} \psi_e^{(+)\dagger} \gamma_0 \gamma^\lambda (1-\gamma_5)\psi_{\nu_e}^{(-)} + \ldots \end{aligned} \tag{10.173}$$

Therefore we obtain, by considering the interaction Hamiltonian density at $t = 0$,

$$\begin{aligned} < f|\mathcal{H}_W|i> &= \frac{G_F}{\sqrt{2}} < e\,\nu_\mu \bar{\nu}_e | J_{\mu\lambda}^\dagger J_e^\lambda |\mu> \\ &= \frac{G_F}{\sqrt{2}} \sqrt{\frac{m_\mu}{E_\mu}} \sqrt{\frac{m_e}{E_e}} \sqrt{\frac{m_{\nu_e}}{E_{\bar{\nu}_e}}} \sqrt{\frac{m_{\nu_\mu}}{E_{\nu_\mu}}} \frac{1}{V^2} e^{i(p_{\nu_\mu}+p_{\bar{\nu}_e}+p_e-p_\mu)x}|_{x^0=0} \\ &\quad u_{r_e}^\dagger(\mathbf{p}_e)\gamma_0\gamma_\lambda(1-\gamma_5) v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e}) u_{r_{\nu_\mu}}^\dagger(\mathbf{p}_{\nu_\mu})\gamma_0\gamma^\lambda(1-\gamma_5) u_{r_\mu}(\mathbf{p}_\mu) \\ &= \frac{G_F}{\sqrt{2}} \sqrt{\frac{m_\mu}{E_\mu}} \sqrt{\frac{m_e}{E_e}} \sqrt{\frac{m_{\nu_e}}{E_{\bar{\nu}_e}}} \sqrt{\frac{m_{\nu_\mu}}{E_{\nu_\mu}}} \frac{1}{V^2} e^{i(p_{\nu_\mu}+p_{\bar{\nu}_e}+p_e-p_\mu)x}|_{x^0=0} \\ &\quad \bar{u}_{r_{n_e}}(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5) v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e}) \bar{u}_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5) u_{r_\mu}(\mathbf{p}_\mu) \end{aligned}$$

and finally

$$\begin{aligned} V_{fi} &= \int d^3x < f|\mathcal{H}_W|i > \\ &= \frac{G_F}{\sqrt{2}}\sqrt{\frac{m_\mu}{\mu}}\sqrt{\frac{m_e}{E_e}}\sqrt{\frac{m_{\nu_e}}{E_{\bar{\nu}_e}}}\sqrt{\frac{m_{\nu_\mu}}{E_{\nu_\mu}}}\frac{1}{V^2}V\delta^3(p_{\nu_\mu}+p_{\bar{\nu}_e}+p_e-p_\mu) \\ & \quad \bar{u}_{r_{ne}}(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e})\bar{u}_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)u_{r_\mu}(\mathbf{p}_\mu) \\ &\equiv \mathcal{M}_{fi}\frac{G_F}{\sqrt{2}}\sqrt{\frac{m_\mu}{E_\mu}}\sqrt{\frac{m_e}{E_e}}\sqrt{\frac{m_{\nu_e}}{E_{\bar{\nu}_e}}}\sqrt{\frac{m_{\nu_\mu}}{E_{\nu_\mu}}}\frac{1}{V}\delta^3(p_{\nu_\mu}+p_{\bar{\nu}_e}+p_e-p_\mu) \end{aligned} \tag{10.174}$$

where

$$\mathcal{M}_{fi} = \bar{u}_{r_{\nu_e}}(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)v_{r_{\nu_e}}(\mathbf{p}_{\bar{\nu}_e})\bar{u}_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)u_{r_\mu}(\mathbf{p}_\mu) \tag{10.175}$$

Let us now compute the decay rate, using the Fermi golden rule and averaging over the initial spin and summing over the final spin

$$\begin{aligned} dw &= \frac{1}{2}\sum_{r_i,r_f} 2\pi\delta(E_f-E_i)|\mathcal{M}_{fi}|^2|\frac{1}{V}\sqrt{\frac{m_\mu}{E_\mu}}\sqrt{\frac{m_e}{E_e}}\sqrt{\frac{m_{\nu_e}}{E_{\bar{\nu}_e}}}\sqrt{\frac{m_{\nu_\mu}}{E_{\nu_\mu}}}|^2 \\ & \quad \frac{(2\pi)^3}{V}\delta^3(p_{\nu_\mu}+p_{\bar{\nu}_e}+p_e-p_\mu)\frac{V}{(2\pi)^3}d^3p_e\frac{V}{(2\pi)^3}d^3p_{\bar{\nu}_e}\frac{V}{(2\pi)^3}d^3p_{\nu_\mu} \end{aligned} \tag{10.176}$$

Therefore the final result can be written as

$$dw = (2\pi)^4\delta^4(P_f-P_i)\frac{m_\mu}{E_\mu}\Pi_f\frac{d^3p_f}{(2\pi)^3}\frac{m_f}{E_f}\frac{1}{2}\sum_{r_i,r_f}|\mathcal{M}_{fi}|^2 \tag{10.177}$$

with (see Appendix H)

$$\frac{1}{2}\sum_{r_i,r_f}|\mathcal{M}_{fi}|^2 = 64G_F^2\frac{1}{2m_{\nu_\mu}2m_{\nu_e}2m_\mu 2m_e}(p_\mu\cdot p_{\bar{\nu}_e})(p_{\nu_\mu}\cdot p_e) \tag{10.178}$$

This can be rewritten as

$$dw = (2\pi)^4\delta^4(P_f-P_i)\Pi_f\frac{d^3p_f}{2E_f(2\pi)^3}\frac{1}{2E_\mu} < M >^2 \tag{10.179}$$

with

$$< M >^2 = 64G_F^2(p_\mu \cdot p_{\bar{\nu}_e})(p_{\nu_\mu} \cdot p_e) \tag{10.180}$$

The total width is computed in the rest frame of the muon

$$p_\mu = (m_\mu, \mathbf{0}) \tag{10.181}$$

In this frame we have

$$(p_\mu \cdot p_{\bar{\nu}_e}) = m_\mu E_{\bar{\nu}_e} \tag{10.182}$$

Furthermore from

$$(p_\mu - p_{\bar{\nu}_e})^2 = (p_{\nu_\mu} + p_e)^2 \tag{10.183}$$

neglecting neutrino and electron masses we get

$$(p_{\nu_\mu} \cdot p_e) \sim \frac{1}{2}m_\mu^2 - m_\mu E_{\bar{\nu}_e} \tag{10.184}$$

and therefore

$$< M >^2 \sim 64G_F^2 \frac{m_\mu^2}{2} E_{\bar{\nu}_e}(m_\mu - 2E_{\bar{\nu}_e}) \tag{10.185}$$

The first integration over $\mathbf{p}_{\nu_\mu}$ is trivial and we obtain

$$\int_{\mathbf{p}_{\nu_\mu}} d\Gamma = \frac{1}{16(2\pi)^5 m_\mu} \frac{d^3 p_{\bar{\nu}_e}}{E_{\bar{\nu}_e}} \frac{d^3 p_e}{E_e E_{\nu_\mu}} \delta(m_\mu - E_{\bar{\nu}_e} - E_{\nu_\mu} - E_e) < M >^2 \tag{10.186}$$

with $< M >^2$ given in (10.185). In the μ rest frame we have $\mathbf{p}_\mu = 0$, $\mathbf{p}_{\nu_\mu} = -\mathbf{p}_{\nu_e} - \mathbf{p}_e$. Then from

$$E_{\nu_\mu} = |\mathbf{p}_{\nu_\mu}| = |\mathbf{p}_{\nu_e} + \mathbf{p}_e| \tag{10.187}$$

we get

$$E_{\nu_\mu} = \sqrt{E_{\bar{\nu}_e}^2 + E_e^2 + 2E_{\bar{\nu}_e} E_e \cos\theta} \tag{10.188}$$

Also we have

$$d^3\mathbf{p}_{\bar{\nu}_e} = E_{\bar{\nu}_e}^2 dE_{\bar{\nu}_e} \sin\theta d\theta d\phi \tag{10.189}$$

To perform the integral over θ we can transform from θ to

$$u \equiv E_{\nu_\mu} = \sqrt{E_{\bar{\nu}_e}^2 + E_e^2 + 2E_{\bar{\nu}_e} E_e \cos\theta} \tag{10.190}$$

So

$$du = -\frac{E_{\bar{\nu}_e} E_e \sin\theta d\theta}{E_{\nu_\mu}} \tag{10.191}$$

and we get

$$\int_{\mathbf{p}_{\nu_e}, \mathbf{p}_{\nu_\mu}} d\Gamma = \int dE_{\bar{\nu}_e} < M >^2 d^3 p_e \frac{1}{16(2\pi)^4 m_\mu E_e^2} \int_{u(\cos\theta=-1)}^{u(\cos\theta=1)} du \delta(m_\mu - E_{\bar{\nu}_e} - u - E_e) \tag{10.192}$$

Let us discuss the extrema of the integration

$$u(\cos\theta = 1) = |E_{\bar{\nu}_e} + E_e| \tag{10.193}$$

$$u(\cos\theta = -1) = |E_{\bar{\nu}_e} - E_e| \tag{10.194}$$

So the integral is different from zero when

$$|E_{\bar{\nu}_e} - E_e| < u = m_\mu - E_{\bar{\nu}_e} - E_e < |E_{\bar{\nu}_e} + E_e| \tag{10.195}$$

which are equivalent to

$$|E_{\bar{\nu}_e} - E_e| + E_{\bar{\nu}_e} + E_e < \frac{m_\mu}{2} < E_{\bar{\nu}_e} + E_e \tag{10.196}$$

This implies

$$E_{\bar{\nu}_e} < \frac{m_\mu}{2} < E_{\bar{\nu}_e} + E_e \tag{10.197}$$

when $E_{\bar{\nu}_e} > E_e$ and

$$E_e < \frac{m_\mu}{2} < E_{\bar{\nu}_e} + E_e \tag{10.198}$$

when $E_{\bar{\nu}_e} < E_e$. Therefore the extrema are

$$\frac{m_\mu}{2} - E_e < E_{\bar{\nu}_e} < \frac{m_\mu}{2} \tag{10.199}$$

By integrating over $dE_{\bar{\nu}_e}$

$$\begin{aligned} \int d\Gamma &= \int_{\frac{m_\mu}{2} - E_e}^{\frac{m_\mu}{2}} dE_{\bar{\nu}_e} < M >^2 d^3 p_e \frac{1}{16(2\pi)^4 m_\mu E_e^2} \\ &= 2G_F^2 \frac{d^3 p_e}{(2\pi)^4} (\frac{m_\mu}{2} - \frac{2E_e}{3}) \end{aligned} \tag{10.200}$$

Then we can integrate over $\mathbf{p}_e$.

$$\Gamma = \frac{2G_F^2}{(2\pi)^4} m_\mu \int_0^{m_\mu/2} dE_e E_e^2 \int_\Omega \sin\theta d\theta (\frac{m_\mu}{2} - \frac{2E_e}{3}) d\phi \tag{10.201}$$

The final result is

$$\Gamma(\mu \to e^- \bar{\nu}_e \nu_\mu) = \frac{G_F^2 m_\mu^5}{3 \times 2^6 \pi^3} \tag{10.202}$$

The decay time is given by

$$\tau_\mu = \frac{3 \times 2^6 \pi^3}{m_\mu^5 G_F^2} \tag{10.203}$$

Using the experimental determination[18] $\tau_\mu \sim 2.2 \times 10^{-6}$sec and $m_\mu \sim 105.7$ MeV, one can extract the value of the Fermi constant

$$G_F = 1.16 \times 10^{-5} \text{GeV}^{-2} \tag{10.204}$$

[18]From [18] $\tau_\mu = (2.1969811 \pm 0.0000022 \times 10^{-6})$ sec, $m_\mu = (105.6583745 \pm 0.0000024)$ MeV . The first measurements of the μ life time were performed by F. Rasetti (1941), $\tau_\mu = (1.5 \pm 0.3 \times 10^{-6})$ sec, and B. Rossi (1943), $\tau_\mu = (2.15 \pm 0.07 \times 10^{-6})$ sec

11 Superfluidity

In my recent studies on liquid helium close to the absolute zero, I have succeeded in discovering a number of new phenomena. . . I am planning to publish part of this work. . . but to do this I need theoretical help. In the Soviet Union it is Landau who has the most perfect command of the theoretical field I need, but unfortunately he has been in custody for a whole year. All this time I have been hoping that he would be released because, frankly speaking, I am unable to believe that he is a state criminal. . ." P. Kapitza, letter to Molotov on April 6,1939

11.1 Brief introduction to superfluids

There are two stable isotopes of the helium: the first He^4 was discovered in 1871 in the solar spectrum, while the He^3 was discovered in 1939 at the Berkeley cyclotron by Louis Alvarez[19].

The isotope He^4, which has a nucleus composed of two protons and two neutrons, is a boson (spin 0) while He^3 composed of two protons and one neutron, is a fermion (spin 1/2). Both liquids have, at low temperatures, low densities and apparently remain liquid down to absolute zero temperature unless a high pressure is applied (25 atm for the He^4). The density of the He^4, at $T = 2.17$ °K and $p = 0.0497$ atm, is

$$\rho_4 = 0.145 \text{ g/cm}^3 \tag{11.1}$$

This property of not freezing comes from the extremely weak Van der Walls forces between the atoms with respect to the quantum fluctuations.

The two liquids behave in different way because the Pauli principle keeps He^3 fermions far each other. The He^4, below T_λ =2.17 °K, enters in a new phase, HeII, Fig. 1.

The first researcher, who liquified the Helium below the gas liquid transition at 4.2 °K, was Kamerlingh Onnes[20] (1908), in the experiments leading to discover superconductivity. Later in 1927, M. Wolfke and W.H. Keesom discovered a new phase transition for the Helium at 2.17 °K, that manifested itself in a discontinuity of the specific heat. After this observation, it took ten years to understand that the new phase was a superfluid phase: the fluid can flow without any friction and viscosity.

The transition line λ is the separation between HeI and HeII phases, the first a normal liquid, the second superfluid. In this phase the liquid is capable of flowing through narrow capillaries without friction. Experiments to prove superfluidity where first performed by Kapitza[21] (1938) at the Institute of Physical Problems in Moscow and indipendently by J.F. Allen and A.D. Misener (1938) in Cambridge.

[19] L. Alvarez 1911-1988, Nobel prize in Physics in 1968 for his contribution to particle physics, in particular the bubble chamber

[20] H. Kamerlingh Onnes, 1853-1926, Nobel prize in Physics in 1913

[21] P.L. Kapitza, 1894-1984, Nobel prize in Physics in 1978

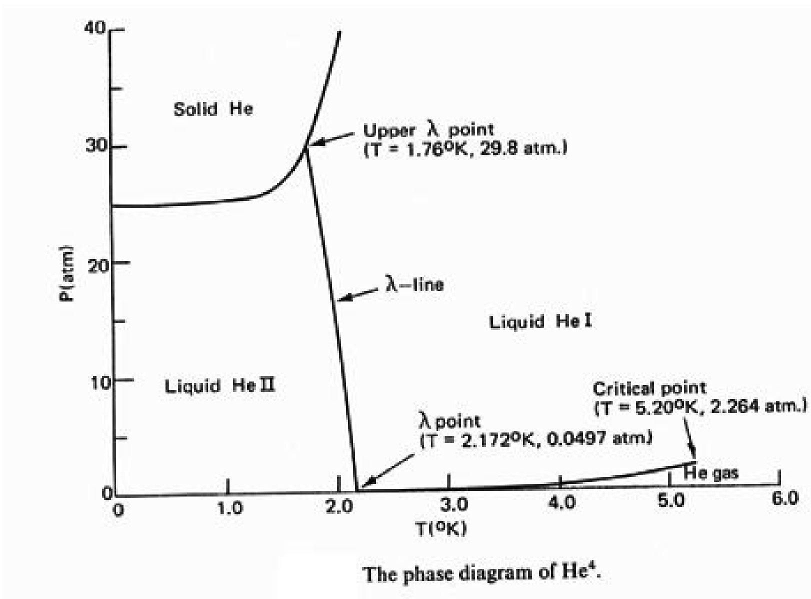

Figure 1: Phase diagram of superfluid He-4 (from J.C.Davis Group, Cornell)

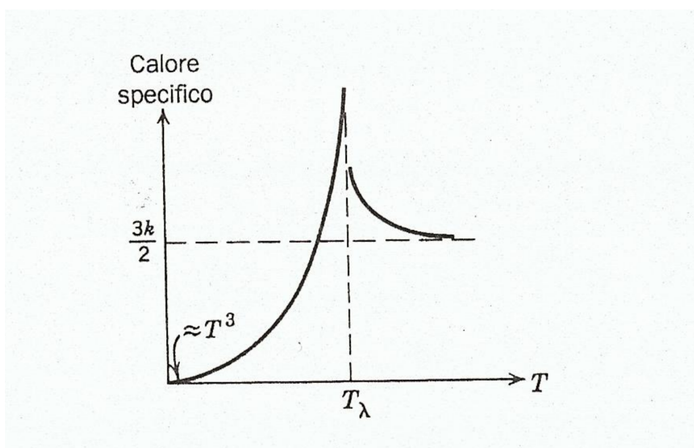

Figure 2: Specific heat (from Huang)

It is natural to associate the λ transition to the Bose Einstein condensation (London, 1938) modified by the molecular interactions. In fact, as we will review in the following, the critical temperature for the condensation in a non interacting gas of bosons is 3.14 °K, very close to T_λ.

The He^3 also becomes superfluid but only at temperatures of the order 10^{-3}°K (1972, Lee, Osheroff and Richardson[22]): in this case pairs of fermions condensate, a similar mechanism to the Cooper pairs for superconductors. Recently, in 2005, the superfluidity has been observed in ultracold Fermi gases (Ketterle[23] et al) at very low temperature, 200 nK.

H^2 solidifies at higher temperature because of stronger interactions among his molecules.

An additional property of the Helium is that at low temperature ($T << T_c$) the specific heat varies as T^3, as shown in Fig. 2.

To explain such a behavior Landau[24] (1941) suggested to interpret the quantum states of the liquid as a phonon gas with the linear dispersion relation

$$\epsilon_k = \hbar c k \tag{11.2}$$

where c is the velocity of propagation of the sound in the fluid. The main idea is that the body moving in the liquid excites sound waves which are collective motions in the liquid. The liquid at low temperature must be treated as a quantum liquid and its excitations are phonons as for the quantum excitations of the vibrations of a crystal. A body moving in the helium at low temperature does not loose energy transferring to single atoms but excites collective quanta as phonons. The Helium dispersion relation curve is measured by neutron scattering, see Fig. 3. Approximately one has

$$\epsilon_k = \hbar c k, \quad k << k_0 \tag{11.3}$$

with $c = (239 \pm 5)$ m/s

$$\epsilon_k = \Delta + \frac{\hbar^2 (k - k_0)^2}{2\sigma}, \quad k \sim k_0 \tag{11.4}$$

[22]Nobel prize in Physics in 1978
[23]W. Ketterle, 1957- Nobel prize in Physics in 2001
[24]L. Landau, 1908-1968, Nobel prize in Physics in 1962

and with $\Delta/k_B = (8.65 \pm 0.04)$ °K, with k_B the Boltzmann constant, $k_0 = (1.92 \pm 0.01)10^8$ cm^{-1}, $\sigma/m = 0.16 \pm 0.01$ with m the mass of the helium atom. Therefore the dispersion relation is linear for small k and has a local minimum at $k = k_0$.

The nature of the excitations in the helium is studied by measuring the change in energy and momentum in cold neutron scattering on Helium superfluid (see for example, Palewsky et al, Physical Review **112**, (1958), 11). The reason is that cold neutrons have momentum close to the momentum of the excitations (the energy of the neutron is ~ 50°K).

Making use of the conservation laws

$$\frac{1}{2m}(\mathbf{p}_i^2 - \mathbf{p}_f^2) = \epsilon(k) \tag{11.5}$$

$$\mathbf{p}_i - \mathbf{p}_f = \hbar k \tag{11.6}$$

where $p_{i(f)}$ are the momenta of the incoming (outgoing) neutron one can obtain the spectrum of the excitations.

Let us now see how it is possible that an object can move in a superfluid without loosing energy (*Landau criterion for superfluidity).* Let us consider an object of mass M moving in a superfluid. The only way in which it can loose energy is by emission of a phonon

$$\frac{\mathbf{P}^2}{2M} - \frac{(\mathbf{P} - \hbar\mathbf{k})^2}{2M} = -\frac{\hbar^2\mathbf{k}^2}{2M} + \frac{\mathbf{P}\cdot\mathbf{k}\hbar}{M} = \epsilon_k \tag{11.7}$$

Therefore

$$(\mathbf{V}\cdot\mathbf{k})\hbar = \epsilon_k + \frac{\hbar^2 k^2}{2M} \geq c\hbar k \tag{11.8}$$

or

$$\hbar V k \geq |\mathbf{V}\cdot\mathbf{k}| \geq ck \tag{11.9}$$

implying

$$V \geq c \tag{11.10}$$

Therefore if $V \leq c$ the process is prohibited. This explanation depends in an essential way from the linear spectrum of the phonons. If the spectrum of the excitations is quadratic the minimum threshold for loosing energy is zero.

At energies around k_0 the object looses energy by emitting the so-called rotons:

$$(\mathbf{V} \cdot \mathbf{k_0})\hbar = \epsilon_{k_0} + \frac{\hbar^2 k_0^2}{2M} \geq \epsilon_{k_0} \tag{11.11}$$

or

$$V k_0 \hbar \geq |V k_0 \hbar \cos\theta| \geq \epsilon_{k_0} = \Delta \tag{11.12}$$

or

$$V \geq v_c \tag{11.13}$$

with $v_c = \Delta/(\hbar k_0) = 8.65\ 1.38\ 10^{-23}\mathrm{J}/(1.055\ 10^{-34}\ 1.92\ \mathrm{Js}) \times 10^{-8}\ \mathrm{cm} \sim 58$ m/s. The rotons are the elementary excitations associated to vortices in the fluid. At low temperature the roton effects are negligible due to the Boltzmann factor $\exp(-\Delta/k_B T)$. At the thermal equilibrium elementary excitations have energies close to the minimum of ϵ that is zero. In presence of a purely phonon spectrum the critical velocity is $c = 239$ m/sec, when rotons are included the critical velocity drops to $v_c = 58$ m/sec (observed in He^4 under pressure).

For a general spectrum ϵ_k the condition is

$$V \geq \frac{\epsilon_k}{\hbar k} \tag{11.14}$$

Therefore the critical velocity, defined as

$$V_c = min(\frac{\epsilon_k}{\hbar k}) \tag{11.15}$$

for a free particle spectrum, $\epsilon(p) = \hbar^2 k^2/2m$ is zero.

11.2 Bose Einstein condensation for an ideal gas

Since the He^4 atom contains six spin 1/2 particles which are bound in such a way that the total spin is zero, we can consider an ideal gas consisting of N bosons in a volume V. The partition function is given by (see. Appendix I)

$$\Omega_B = k_B T \sum_{\mathbf{p}} \log\left[1 - \exp\left[\beta(-\epsilon_{\mathbf{p}} + \mu)\right]\right] \tag{11.16}$$

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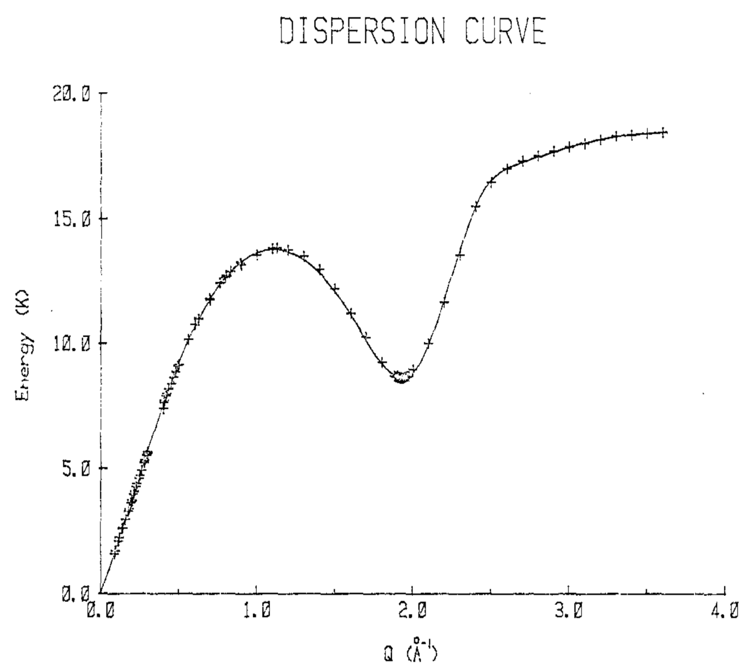

Fig. 1. Least squares spline fit to the neutron scattering data of Table I. The experimental data of Table I are represented as crosses.

Figure 3: $E(°\mathrm{K})$ versus $k(\mathrm{angstrom}^{-1})$. Experimental data and fit to the neutron scattering data, From R.J. Donnelly, J.A. Donnelly and R.N. Hills, Journal of Low temperature Physics, **44** 1981 471

where μ is the chemical potential and where $\beta = 1/k_B T$. Let us now recall the average number of an ideal gas of N non interacting bosons, derived by using the grand partition function,

$$N_B = -\frac{\partial \Omega}{\partial \mu} = \sum_{\mathbf{p}} \frac{1}{\exp\left[\beta(\epsilon_{\mathbf{p}} - \mu)\right] - 1} \tag{11.17}$$

and passing to the continuum

$$N = N_0 + \frac{V}{h^3} \int d^3p \frac{1}{\exp\left[\beta(\epsilon_{\mathbf{p}} - \mu)\right] - 1} \tag{11.18}$$

where N_0 is the number of bosons with $\mathbf{p} = 0$.

At $T = 0$ all bosons occupy the state with $p = 0$ *(Bose condensation)*. At finite temperature, $T \neq 0$, only a fraction N_0/N of bosons remain in the state $p = 0$. For $T >> T_c$, where T_c is the critical temperature, there is no condensate, $N_0 = 0$, $n = N/V$ requiring $\mu < 0$ because of the singularity for $\mu > 0$. When T decreases, for fixed N/V, the absolute value of the chemical potential increases until for temperatures sufficiently low reaches the value of 0. The condensation starts when $\mu = 0$

$$\frac{N_0}{N} = 0 \quad \mu = 0 \tag{11.19}$$

or

$$\begin{aligned} N &= 0 + \frac{V}{h^3} \int d^3p \frac{1}{\exp[\frac{\epsilon_{\mathbf{p}}}{k_B T_c}] - 1} \\ &= \frac{V}{h^3} (2mk_B T_c)^{3/2} 4\pi \int_0^\infty x^2 \frac{1}{e^{x^2} - 1} dx \\ &= \frac{V}{\lambda_c^3} g_{3/2}(1) \end{aligned} \tag{11.20}$$

where we have performed the substitution

$$x = \frac{p}{\sqrt{2mk_B T_c}} \tag{11.21}$$

and

$$\lambda_c = \sqrt{\frac{2\pi\hbar^2}{mk_B T_c}} \tag{11.22}$$

From this condition we can get the critical temperature

$$T_c = \frac{1}{k_B} \frac{2\pi\hbar^2/m}{[v g_{3/2}(1)]^{2/3}} = \frac{1}{k_B} \frac{2\pi\hbar^2/m}{[\zeta(3/2)]^{2/3}} (\frac{N}{V})^{2/3} \tag{11.23}$$

where $g_{3/2}(1) = \zeta(3/2) \sim 2.6$ with ζ the Riemann function (see Appendix I). Using the numerical values $\rho_{He4} = 0.145$ g/cm$^3 = m_{He^4} N/V$ with $m_{He^4} = 4m_p$, $m_p = 4 \times 1.67 \times 10^{-27}$Kg, one can get N/V. Then inserting $\hbar = 1.05510^{-34}$ J sec, the Boltzmann constant $k_B = 1.38\ 10^{-23}$J/°K, $\zeta(3/2) = 2.61$, we get $T_c = 3.14$ °K. This temperature is very close to the critical temperature of liquid Helium, $T_\lambda = 2.17$ °K, below which the helium becomes superfluid.

For $T < T_c$ μ remains zero and using eq. (11.18), with $\mu = 0$ we get

$$\begin{aligned} \frac{N - N_0}{V} &= \int \frac{d^3p}{h^3} \frac{1}{\exp[\epsilon_{\mathbf{p}}/k_B T] - 1} \\ &= \frac{1}{\lambda^3} g_{3/2}(1) = \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(1) = \frac{N}{V}\left(\frac{T}{T_c}\right)^{3/2} \end{aligned} \tag{11.24}$$

or

$$\frac{N - N_0}{N} = \left(\frac{T}{T_c}\right)^{3/2} \tag{11.25}$$

or

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \tag{11.26}$$

and

$$\frac{N_0}{V} = \frac{N}{V}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] \quad T < T_c \tag{11.27}$$

In conclusion below T_c a fraction of particles occupy the state with $p = 0$. Therefore for $T < T_c$ we have a condensate with a macroscopic number of particles in the same quantum state with $p = 0$. Bose Einstein condensation provides only a qualitative description of superfluidity. For example the specific heat of a non interacting boson gas vanishes as $T^{3/2}$ while the specific heat of He^4 behaves as T^3. Furthermore we have superfluidity only for zero velocity of the atoms given that the spectrum is the free particle one, $\epsilon = p^2/2m$.

11.3 Quantization of the Schrödinger field

The Schrödinger field $\phi(\mathbf{x}, t)$ satisfies the equation

$$i\hbar\frac{\partial}{\partial t}\phi = -\hbar^2\frac{\nabla^2}{2m}\phi \tag{11.28}$$

which can be derived from the following Lagrangian

$$\mathcal{L} = i\hbar\phi^*\dot{\phi} - \hbar^2\frac{1}{2m}\nabla\phi^*\cdot\nabla\phi \tag{11.29}$$

The corresponding Hamiltonian density is given by

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L} = \hbar^2\frac{1}{2m}\nabla\phi^*\cdot\nabla\phi \tag{11.30}$$

with $\Pi = \partial\mathcal{L}/\partial\dot{\phi} = i\hbar\phi^*$. The commutation relations are

$$[\phi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\hbar\delta^3(x - y) \tag{11.31}$$

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\Pi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = 0 \tag{11.32}$$

or equivalently

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a^\dagger_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = 0 \quad [a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \tag{11.33}$$

Using the general solution of the Schrödinger equation

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_k t)}a_{\mathbf{k}} \tag{11.34}$$

with $\omega_k = \hbar\mathbf{k}^2/2m$, the Hamiltonian of the field is given by

$$H = \int d^3x\mathcal{H} = \sum_{\mathbf{k}} \hbar\omega_k a^\dagger_{\mathbf{k}}a_{\mathbf{k}} \tag{11.35}$$

As an application of the non relativistic field theory we will consider the superfluidity theory.

Bogoliubov (1947) studied the fundamental state of a dilute gas of weakly interacting bosons and their excitations using the second quantization of a many body system and assuming the following interaction Hamiltonian, see [9, 10]:

$$H_I = \frac{1}{2} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}'_1+\mathbf{k}'_2} W(|\mathbf{k}_1 - \mathbf{k}'_1|) a^\dagger_{\mathbf{k}'_1} a^\dagger_{\mathbf{k}'_2} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \tag{11.36}$$

where $\hbar\mathbf{k}_1, \hbar\mathbf{k}_2$ ($\hbar\mathbf{k}'_1, \hbar\mathbf{k}'_2$) represent the momenta of incoming (outgoing) bosons and $\hbar\mathbf{k}'_1 - \hbar\mathbf{k}_1$ the transfer momentum. The function $W(k)$ is the Fourier transform of the four boson interaction

$$W(k) = \int d\vec{r}\, W(r) e^{i\mathbf{k}\cdot\mathbf{r}} \tag{11.37}$$

The dilute gas approximation justifies to consider only two boson scattering.

Bogoliubov was the first to prove the existence of the "phonons" as collective excitations in the quantum liquid.

Before discussing the Bogoliubov approach let us first review how one can describe the superfluidity phase by use of the general approach of Ginzburg[25]-Landau to phase transitions.

11.4 Ginzburg-Landau Model

The superfluidity phase transition can be derived within the Ginzburg-Landau approach to phase transitions, assuming that the states of the system are described by a scalar field which can be interpreted as the wave function of the superfluid.

The Hamiltonian of the model (the free energy), which was proposed as an effective description of field theory for phase transitions, is given by

$$H_{eff} = \int d^3x \mathcal{H}_{eff} \tag{11.38}$$

with

$$\mathcal{H}_{eff} = \frac{\hbar^2}{2m} (\nabla\phi)^\dagger (\nabla\phi) - \mu\phi^\dagger\phi + \frac{1}{2} g (\phi^\dagger\phi)^2 \tag{11.39}$$

with μ and g positive constants. In particular $\mu < 0$ for $T > T_c$ and passes through zero at T_c.

[25] V.L. Ginzburg, 1916-2009, Nobel prize in Physics in 2003

Therefore the potential can be identified as

$$V(\phi) = -\mu\phi^\dagger\phi + \frac{1}{2}g(\phi^\dagger\phi)^2 \tag{11.40}$$

and is dominated for low density by the chemical potential and at large density by the g term. The form of the potential is shown in Fig. 4. A microscopic interpretation of the parameters μ and g can be found in [10]. They can be related to the strength of the four boson interaction and to the density of the condensate (see next Section).

Let us now assume that the chemical potential depends on the temperature, so that $\mu < 0$ for $T > T_c$, T_c being the critical temperature, and $\mu > 0$ for $T < T_c$. The potential, for $T < T_c$, has a maximum in $|\phi| = 0$ and a minimum for

$$\phi^\dagger\phi = \frac{\mu}{g} \tag{11.41}$$

or

$$\phi(x) = \phi_0 \exp{(i\psi)} = \sqrt{\frac{\mu}{g}} \exp{(i\psi)}, \tag{11.42}$$

Let us notice that the Hamiltonian (11.39) is invariant under the transformation

$$\phi(x) \to \phi(x) \exp{(i\alpha)}, \quad \alpha \in [0, 2\pi) \tag{11.43}$$

while the minimum state is not ($\phi_0 \exp{(i\psi)} \to \phi_0 \exp{[i(\psi + \alpha)]}$). This phenomenon is called *Spontaneous Symmetry Breaking* and it is used for describing phase transitions in different domains of physics.

The minimum is degenerate varying $\psi \in [0, 2\pi)$. For simplicity let us choose the minimum at $\psi = 0$. The series of the field in normal modes can be performed with respect to the new minimum in ϕ_0

$$\phi(x) = \phi_0 + \tilde{\phi}(x) = \phi_0 + \sum_{\mathbf{k}\neq 0} \frac{1}{\sqrt{V}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{11.44}$$

Notice that we are working in the Schrödinger representation: the operator ϕ is evaluated at $t = 0$.

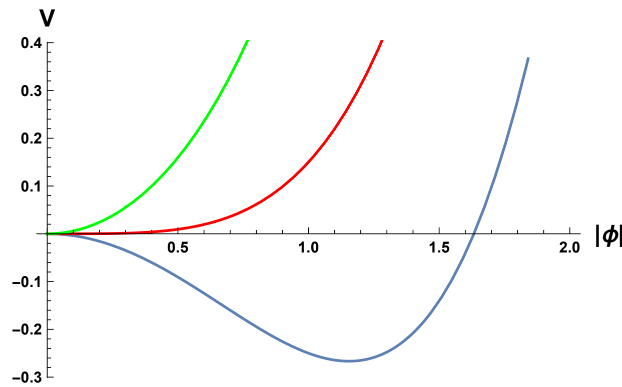

Figure 4: The potential V corresponding to eq.(11.40) as a function of $|\phi| = \sqrt{\phi^\dagger\phi}$ for $T < T_c$ (blue), $T = T_c$ (red) and $T > T_c$ (green) for given values of μ and g and in suitable units.

By substituting eq.(11.44) in the potential (11.40) one gets, by expanding to second order in $\tilde{\phi}$

$$\begin{aligned} V &= -\mu\Big[\phi_0^2 + \phi_0(\tilde{\phi} + \tilde{\phi}^\dagger) + \tilde{\phi}^\dagger\tilde{\phi}\Big] + \frac{1}{2}g\Big[\phi_0^4 + \phi_0^2(\tilde{\phi} + \tilde{\phi}^\dagger)^2 + 2\phi_0^3(\tilde{\phi} + \tilde{\phi}^\dagger) + 2\phi_0^2\tilde{\phi}^\dagger\tilde{\phi}\Big] \\ &\quad + \mathcal{O}(\tilde{\phi}^3, \tilde{\phi}^4) \\ &= -g\phi_0^2\Big[\phi_0^2 + \phi_0(\tilde{\phi} + \tilde{\phi}^\dagger) + \tilde{\phi}^\dagger\tilde{\phi}\Big] + \frac{1}{2}g\phi_0^4 + \frac{1}{2}g\phi_0^2(\tilde{\phi} + \tilde{\phi}^\dagger)^2 + g\phi_0^3(\tilde{\phi} + \tilde{\phi}^\dagger) \\ &\quad + g\phi_0^2\tilde{\phi}^\dagger\tilde{\phi} + \mathcal{O}(\tilde{\phi}^3, \tilde{\phi}^4) \\ &= -\frac{1}{2}g\phi_0^4 + \frac{1}{2}g\phi_0^2(\tilde{\phi} + \tilde{\phi}^\dagger)^2 + \mathcal{O}(\tilde{\phi}^3, \tilde{\phi}^4) \\ &= -\frac{\mu^2}{2g} + \frac{1}{2}\mu(\tilde{\phi} + \tilde{\phi}^\dagger)^2 + \mathcal{O}(\tilde{\phi}^3, \tilde{\phi}^4) \end{aligned} \tag{11.45}$$

Let us now quantize the scalar field $\tilde{\phi}$, by requiring standard commutation relations $a_{\mathbf{k}}$ e $a_{\mathbf{k}}^\dagger$. By substituting the normal mode series and integrating in d^3x the Hamiltonian density, one obtains

$$H_{eff} = \sum_{\mathbf{k}\neq 0}\left[\left(\mu + \frac{\hbar^2\mathbf{k}^2}{2m}\right)a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{\mu}{2}(a_{\mathbf{k}}a_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger)\right] + E_0 \tag{11.46}$$

where

$$E_0 = -\frac{\mu^2}{2g}V + \frac{\mu}{2}\sum_{\mathbf{k}\neq 0} 1 \tag{11.47}$$

In eq.(11.47), V denotes the space volume.

The Hamiltonian (11.46) is not diagonal in the basis of occupation numbers because of the bi-linear terms a e in $a^\dagger$. However it is possible to find a transformation (Bogoliubov transformation, see Appendix K) from $a_{\mathbf{k}}(a^\dagger_{\mathbf{k}})$ to the operators $A_{\mathbf{k}}(A^\dagger_{\mathbf{k}})$, defined as

$$A_{\mathbf{k}} = \cosh(\frac{\theta_k}{2})a_{\mathbf{k}} + \sinh(\frac{\theta_k}{2})a^\dagger_{-\mathbf{k}} \tag{11.48}$$

with

$$\tanh\theta_k = \frac{\mu}{\mu + \frac{\hbar^2\mathbf{k}^2}{2m}} \tag{11.49}$$

One has

$$[A_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \tag{11.50}$$

Furthermore

$$H_{eff} = E_0^{new} + \sum_{\mathbf{k}\neq 0}\epsilon(k)A^\dagger_{\mathbf{k}}A_{\mathbf{k}} \tag{11.51}$$

with

$$E_0^{new} = E_0 - \sum_{\mathbf{k}\neq 0}\epsilon(k)\sinh^2(\frac{\theta_k}{2}) \tag{11.52}$$

and

$$\epsilon(k) = \sqrt{\left(\mu + \frac{\hbar^2\mathbf{k}^2}{2m}\right)^2 - \mu^2} = \sqrt{\frac{\mu}{m}\hbar^2\mathbf{k}^2 + \left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^2} \tag{11.53}$$

The fundamental state is defined as

$$A_{\mathbf{k}}|\tilde{\phi}_0 >= 0 \tag{11.54}$$

Starting from this new vacuum state one can build the new Fock space with the operators $A^\dagger_{\mathbf{k}}$. For example, the first excited state (or quasi-particle) is given by

$$A^\dagger_{\mathbf{k}}|\tilde{\phi}_0 > \tag{11.55}$$

with energy

$$\epsilon(k) = \sqrt{\frac{\mu}{m}\hbar^2\mathbf{k}^2 + \left(\frac{\hbar^2\mathbf{k}^2}{2m}\right)^2} \tag{11.56}$$

Therefore the spectrum is linear for small k while for large k behaves as k^2.

In conclusion the Ginzburg-Landau approach is able to explain the spectrum of the liquid helium at low k but does not reproduce the local minimum due to the rotons.

The relation of the new vacuum $|\tilde{\phi}_0>$ with the Fock vacuum state $|0>$ is given in Appendix J.

11.5 Bogoliubov approach

We are assuming the boson gas in a rarefied state or that the average distance between the particles

$$d \sim (\frac{N}{V})^{1/3} >> r_0 \tag{11.57}$$

where r_0 denotes the range of the interaction. Under this hypothesis we can consider only two boson in two boson interactions and neglect scattering involving more than four bosons.[26] The Hamiltonian describing the two boson in two boson interaction is the Bogoliubov Hamiltonian

$$H_I = \frac{1}{2}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_1'+\mathbf{k}_2'} W(|\mathbf{k}_1-\mathbf{k}_1'|)a^\dagger_{\mathbf{k}_1'}a^\dagger_{\mathbf{k}_2'}a_{\mathbf{k}_1}a_{\mathbf{k}_2} \tag{11.58}$$

For $T << T_c$ all bosons tend to belong to the state $k=0$, therefore $N_0 \sim N$, where N is the total number of bosons, and

$$\frac{N-N_0}{N_0} << 1 \tag{11.59}$$

Since $N_0 \sim N >> 1$ one can assume the operator a_0 to be c-number $a_0 \sim a_0^\dagger \sim \sqrt{N_0}$. So we consider the operators $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ *small* with respect to a_0 and $a_0^\dagger$, expanding the interaction Hamiltonian keeping only the terms

[26] We have also a cutoff on the momentum $p < \frac{\hbar}{r_0}$

which are linear or bi-linear in N_0. In other words Bogoliubov separates the condensate in the expansion of the field:

$$\phi \sim \frac{1}{\sqrt{V}}(\sqrt{N_0} + \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}} \exp\left[i(\mathbf{k}\cdot\mathbf{x})\right]) \tag{11.60}$$

Let us first rewrite the interaction Hamiltonian by taking into account the momentum conservation

$$H_I = \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}'_1} W(|\mathbf{k}_1 - \mathbf{k}'_1|) a^\dagger_{\mathbf{k}'_1} a^\dagger_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}'_1} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \tag{11.61}$$

Let us first list all the cases where the indices are zero. One has four zero indices for $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}'_1 = 0$. One has two zero indices when (we list also the transfer momentum)

$$\begin{array}{rl}
\mathbf{k}_1 = \mathbf{k}'_1 = 0 & \mathbf{k}_1 - \mathbf{k}'_1 = 0 \\
\mathbf{k}_1 = \mathbf{k}_2 = 0 & \mathbf{k}_1 - \mathbf{k}'_1 = -\mathbf{k}'_1 \\
\mathbf{k}_2 = \mathbf{k}'_1 = 0 & \mathbf{k}_1 - \mathbf{k}'_1 = \mathbf{k}_1 \\
\mathbf{k}_2 = \mathbf{k}_1 - \mathbf{k}'_1 = 0 & \mathbf{k}_1 - \mathbf{k}'_1 = 0 \\
\mathbf{k}_1 = \mathbf{k}_2 - \mathbf{k}'_1 = 0 & \mathbf{k}_1 - \mathbf{k}'_1 = -\mathbf{k}'_1 \\
\mathbf{k}'_1 = \mathbf{k}_1 + \mathbf{k}_2 = 0 & \mathbf{k}_1 - \mathbf{k}'_1 = \mathbf{k}_1
\end{array} \tag{11.62}$$

Therefore neglecting all terms of order $O(a^3_{\mathbf{k}})$ and $O(a^4_{\mathbf{k}})$ we obtain

$$\begin{aligned}
H_I &= \frac{1}{2}\Big[W(0)(a^\dagger_0 a_0)^2 \\
&+ W(0)a^\dagger_0 a_0 \sum_{\mathbf{k}_2} a^\dagger_{\mathbf{k}_2} a_{\mathbf{k}_2} + a_0 a_0 \sum_{\mathbf{k}'_1} W(k'_1) a^\dagger_{\mathbf{k}'_1} a^\dagger_{-\mathbf{k}'_1} \\
&+ a^\dagger_0 a_0 \sum_{\mathbf{k}_1} W(k_1) a^\dagger_{\mathbf{k}_1} a_{\mathbf{k}_1} + W(0) a^\dagger_0 a_0 \sum_{\mathbf{k}_1} a^\dagger_{\mathbf{k}_1} a_{\mathbf{k}_1} \\
&+ a^\dagger_0 a_0 \sum_{\mathbf{k}'_1} W(k'_1) a^\dagger_{\mathbf{k}'_1} a_{\mathbf{k}'_1} + a^\dagger_0 a^\dagger_0 \sum_{\mathbf{k}_1} W(k_1) a_{-\mathbf{k}_1} a_{\mathbf{k}_1}\Big] \\
&= \frac{1}{2}\Big[W(0)N_0^2 + 2N_0 \sum_{\mathbf{k}\neq 0} (W(0) + W(k)) a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \\
&+ N_0 \sum_{\mathbf{k}\neq 0} W(k)(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}})\Big]
\end{aligned} \tag{11.63}$$

Furthermore the number operator

$$N = a_0^\dagger a_0 + \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{11.64}$$

so that neglecting order $O(a_{\mathbf{k}}^4)$

$$N_0^2 \sim N^2 - 2N \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \tag{11.65}$$

By substituting eq.(11.65) in eq.(11.63) we get

$$\begin{aligned} H_I &= \frac{1}{2}\Big[W(0)N^2 + 2N \sum_{\mathbf{k}\neq 0} W(k) a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \\ &+ N \sum_{\mathbf{k}\neq 0} W(k)(a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} a_{-\mathbf{k}})\Big] \end{aligned} \tag{11.66}$$

The total Hamiltonian, obtained adding to H_I the kinetic term, can be diagonalized as shown in Appendix K by the Bogoliubov transformation given in eq.(K.3)

$$\tanh\theta_k = \frac{NW(k)}{NW(k) + \frac{\hbar^2 k^2}{2m}} \tag{11.67}$$

The total Hamiltonian can be rewritten as

$$\sum_{\mathbf{k}} \epsilon(k) A_{\mathbf{k}}^\dagger A_{\mathbf{k}} \tag{11.68}$$

with

$$\begin{aligned} \epsilon(k) &= \sqrt{\left[NW(k) + \frac{\hbar^2 k^2}{2m}\right]^2 - [NW(k)]^2} \\ &= \sqrt{\left[\frac{\hbar^2 k^2}{2m}\right]^2 + NW(k)\frac{\hbar^2 k^2}{m}} \end{aligned} \tag{11.69}$$

With a suitable choice of $W(k)$ one is able to reproduce not only the phonon part of the spectrum but also the roton part.

Finally, by comparison with the Ginzburg-Landau Hamiltonian, we obtain the identification

$$\mu \sim NW(0), \;\; g = W(0)V \tag{11.70}$$

and so if we neglect the interaction, we recover $\mu = 0$, or the vanishing of the μ parameter below the critical temperature in the free boson gas approach to superfluidity.

12 Superconductivity

12.1 BCS Hamiltonian

Let us now consider, as a second application of non relativistic quantum field theory, the phenomenon of superconductivity. Superconductivity is characterized by two main properties:

- In many metals, for example lead, tin, aluminium, cadmium, niobium... below a critical temperature $T_c \sim$ few $^\circ K$, resistivity drops to zero (the discovery was made working at low temperature with mercury by Kamerlingh Onnes, 1911)
- Meissner effect: exclusion of magnetic fields from superconducting regions. The magnetic field decreases exponentially over distances of order 500 Å (Meissner, Ochsenfeld [27], 1933)

The theoretical explanation is based on the formation of Cooper[28] pairs: below the critical temperature the interaction between electrons close to the Fermi surface and the phonons of the ion lattice can compensate for the Coulomb repulsion and provides the mechanism for the formation of Cooper pairs. Cooper showed (1956) that the Fermi sea of electrons is unstable against the formation of Cooper pairs.

One can show that the excitations of such a system have a spectrum which has a minimum corresponding to a finite energy gap and therefore an electron moving in the metal cannot loose energy if its energy is below the gap. Therefore the current flows without resistivity.

In the following we will follow the Bardeen[29], Cooper, Schrieffer[30] approach (1957). We consider a non relativistic spinor field

$$\psi_\sigma(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} u_\sigma \exp\left[-i(\omega_k t - \mathbf{k} \cdot \mathbf{x})\right] \tag{12.1}$$

[27] F.W. Meissner 1882-1974, R. Ochsenfeld, 1901-1993

[28] L. Cooper, 1930-, Nobel prize in Physics in 1972

[29] John Bardeen, 1908-1991, Nobel prize in Physics in 1956 and in 1972

[30] J. R. Schrieffer, 1931-, Nobel prize in Physics in 1972

where $u_\sigma, \sigma = 1, 2$ are the two orthogonal two dimensional spinors and the operators $c_{\mathbf{k},\sigma}, c^\dagger_{\mathbf{k}',\sigma'}$ satisfy the anticommutation relations

$$[c_{\mathbf{k},\sigma}, c^\dagger_{\mathbf{k}',\sigma'}]_+ = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'} \tag{12.2}$$

The Bardeen, Cooper and Schrieffer (1957) Hamiltonian is given by the grand canonical Hamiltonian which includes a term $-\mu N$ where $\mu \sim \mu_F = p_F^2/2m$. In other words the chemical potential is approximated by its value at the Fermi surface. We have

$$H = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} - \frac{1}{V}\sum_{\mathbf{k},\mathbf{k}'} W_{\mathbf{k},\mathbf{l}} c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}',\uparrow} \tag{12.3}$$

with

$$\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \epsilon_F = \frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k_F^2}{2m} \tag{12.4}$$

and $W_{\mathbf{k}\mathbf{k}'} = W \neq 0$ only for electrons close to the Fermi surface

$$|\xi_{\mathbf{k}}|, |\xi_{\mathbf{k}'}| \leq \hbar\omega_D \tag{12.5}$$

where ω_D is the Debye frequency[31] and $\hbar\omega_D$ can be considered as an estimate of the phonon energy, otherwise $W_{\mathbf{k}\mathbf{k}'} = 0$. This can be understood from the fact that only electrons close to the Fermi states can scatter from a phonon and find a different and not occupied final state. The shell is very tiny $\omega_D/\mu_F \sim 10^{-3}$.

Since at low temperature the phonon electron interaction generates a condensate with pairs of electron of opposite spin and momentum the new vacuum (fundamental state) of the theory must be such that

$$< c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} >\neq 0 \quad and \quad < c^\dagger_{-\mathbf{k},\downarrow} c^\dagger_{\mathbf{k},\uparrow} >\neq 0 \tag{12.6}$$

Therefore the new vacuum $|BCS >\neq |0>$ since the standard vacuum $|0>$ satisfies

$$c_{\mathbf{k},\sigma}|0>= 0 \tag{12.7}$$

[31] The Debye frequency is defined by the total number of phonon modes: $N = \sum \mathbf{k} = V/(2\pi)^3 \int d^3k = V/2\pi^2 v_s^3 \hbar^3 \int_0^{\hbar\Omega_D} \epsilon^2 d\epsilon = V\omega_D^3/6\pi^2 v_s^3$ where we have made use of the dispersion relation $\epsilon = v_s \hbar k$

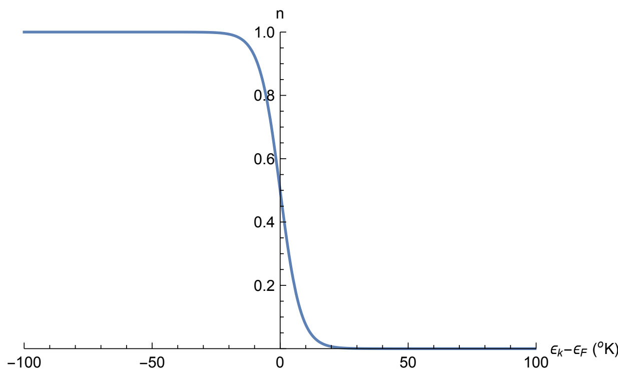

Figure 5: The distribution function of electrons at 4 °K, as a function of $\epsilon_{\mathbf{k}} - \epsilon_F$. $\hbar\omega_D$ is of the order of 10^2 °K.

Let us now see whether it is possible to find a new vacuum $|BCS>$ such that

$$< BCS| : c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} : |BCS >= 0 \tag{12.8}$$

and

$$< BCS| c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} |BCS >= Q_k \neq 0 \tag{12.9}$$

or

$$c_{-\mathbf{k},\uparrow} c_{\mathbf{k},\downarrow} = Q_{\mathbf{k}} + : c_{-\mathbf{k},\uparrow} c_{\mathbf{k},\downarrow} : \tag{12.10}$$

As we have done in superfluidity we perform the transformation from $c_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}$ to a new pair of operators $A_{\mathbf{k}}, B_{\mathbf{k}}$

$$\begin{aligned} A_{\mathbf{k}} &= u_{\mathbf{k}} c_{\mathbf{k},\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k},\downarrow} \\ B_{\mathbf{k}} &= u_{\mathbf{k}} c_{-\mathbf{k},\downarrow} + v_{\mathbf{k}} c^\dagger_{\mathbf{k},\uparrow} \end{aligned} \tag{12.11}$$

where we assume $u_{\mathbf{k}}, v_{\mathbf{k}}$ real.

By requiring the anticommutation relation for $A_{\mathbf{k}}, B_{\mathbf{k}}$

$$[A_{\mathbf{k}}, A^\dagger_{\mathbf{k}'}]_+ = [B_{\mathbf{k}}, B^\dagger_{\mathbf{k}'}]_+ = \delta_{\mathbf{k},\mathbf{k}'} \tag{12.12}$$

we get

$$u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1 \tag{12.13}$$

which can be satisfied assuming

$$u_{\mathbf{k}} = \cos\theta_{\mathbf{k}}\,, \quad v_{\mathbf{k}} = \sin\theta_{\mathbf{k}} \tag{12.14}$$

We require also

$$A_{\mathbf{k}}|BCS>=B_{\mathbf{k}}|BCS>=0 \tag{12.15}$$

By using the inverse relations

$$\begin{aligned} c_{\mathbf{k},\uparrow} &= u_{\mathbf{k}}A_{\mathbf{k}} + v_{\mathbf{k}}B^{\dagger}_{\mathbf{k}} \\ c_{-\mathbf{k},\downarrow} &= -v_{\mathbf{k}}A^{\dagger}_{\mathbf{k}} + u_{\mathbf{k}}B_{\mathbf{k}} \end{aligned} \tag{12.16}$$

we get

$$c_{-\mathbf{k},\downarrow}c_{\mathbf{k},\uparrow} = (-v_{\mathbf{k}}A^{\dagger}_{\mathbf{k}} + u_{\mathbf{k}}B_{\mathbf{k}})(u_{\mathbf{k}}A_{\mathbf{k}} + v_{\mathbf{k}}B^{\dagger}_{\mathbf{k}}) = u_{\mathbf{k}}v_{\mathbf{k}} + :c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}: \tag{12.17}$$

with

$$\begin{aligned} :c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}: &= -u_{\mathbf{k}}v_{\mathbf{k}}(A^{\dagger}_{\mathbf{k}}A_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}}B_{\mathbf{k}}) \\ &+ u^2_{\mathbf{k}}B_{\mathbf{k}}A_{\mathbf{k}} - v^2_{\mathbf{k}}A^{\dagger}_{\mathbf{k}}B^{\dagger}_{\mathbf{k}} \end{aligned} \tag{12.18}$$

We can now perform the transformation in the Hamiltonian: first the kinetic term

$$\begin{aligned} \sum_{\mathbf{k}}\xi_{\mathbf{k}}(c^{\dagger}_{\mathbf{k}\uparrow}c_{\mathbf{k}\uparrow} + c^{\dagger}_{\mathbf{k}\downarrow}c_{\mathbf{k}\downarrow}) &= 2\sum_{\mathbf{k}}\xi_{\mathbf{k}}v^2_{\mathbf{k}} + \sum_{\mathbf{k}}\xi_{\mathbf{k}}(u^2_{\mathbf{k}} - v^2_{\mathbf{k}})(A^{\dagger}_{\mathbf{k}}A_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}}B_{\mathbf{k}}) \\ &- 2\sum_{\mathbf{k}}\xi_{\mathbf{k}}u_{\mathbf{k}}v_{\mathbf{k}}(B^{\dagger}_{\mathbf{k}}A^{\dagger}_{\mathbf{k}} + A_{\mathbf{k}}B_{\mathbf{k}}) \end{aligned} \tag{12.19}$$

and then the interaction term, neglecting terms of order $O(c^4_{\mathbf{k}})$,

$$\begin{aligned} -\frac{1}{V}\sum_{\mathbf{k},\mathbf{k}'}W_{\mathbf{k}\mathbf{k}'}c^{\dagger}_{\mathbf{k},\uparrow}c^{\dagger}_{-\mathbf{k},\downarrow}c_{-\mathbf{k}',\downarrow}c_{\mathbf{k}',\uparrow} &= -\frac{1}{V}\sum_{\mathbf{k},\mathbf{k}'}W_{\mathbf{k}\mathbf{k}'}u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k}'}v_{\mathbf{k}'} \\ &- \frac{1}{V}\sum_{\mathbf{k}}u_{\mathbf{k}}v_{\mathbf{k}}\sum_{\mathbf{k}'}W_{\mathbf{k}\mathbf{k}'}(:c^{\dagger}_{\mathbf{k}',\uparrow}c^{\dagger}_{-\mathbf{k}',\downarrow}: + :c_{-\mathbf{k}',\downarrow}c_{\mathbf{k}',\uparrow}:) \end{aligned} \tag{12.20}$$

Summing eq.(12.19) and eq.(12.20) we get

$$\begin{aligned} &2\sum_{\mathbf{k}}\xi_{\mathbf{k}}v^2_{\mathbf{k}} + \sum_{\mathbf{k}}\xi_{\mathbf{k}}(u^2_{\mathbf{k}} - v^2_{\mathbf{k}})(A^{\dagger}_{\mathbf{k}}A_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}}B_{\mathbf{k}}) \\ - &2\sum_{\mathbf{k}}\xi_{\mathbf{k}}u_{\mathbf{k}}v_{\mathbf{k}}(A_{\mathbf{k}}B_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}}A^{\dagger}_{\mathbf{k}}) \end{aligned}$$

$$
\begin{aligned}
- & \frac{1}{V}[\sum_{\mathbf{kk}'} W_{\mathbf{kk}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} \\
+ & \sum_{\mathbf{k},\mathbf{k}'} W_{\mathbf{kk}'}(-2) u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} (A^{\dagger}_{\mathbf{k}'} A_{\mathbf{k}'} + B^{\dagger}_{\mathbf{k}'} B_{\mathbf{k}'}) \\
+ & \sum_{\mathbf{k},\mathbf{k}'} W_{\mathbf{kk}'} u_{\mathbf{k}} v_{\mathbf{k}} (u^2_{\mathbf{k}'} - v^2_{\mathbf{k}'})(B_{\mathbf{k}'} A_{\mathbf{k}'} + A^{\dagger}_{\mathbf{k}'} B^{\dagger}_{\mathbf{k}'})]
\end{aligned} \tag{12.21}
$$

Requiring the vanishing of the term $A_{\mathbf{k}} B_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}} A^{\dagger}_{\mathbf{k}}$ one gets

$$
2\xi_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}} = \frac{1}{V} \sum_{\mathbf{k}'} W_{\mathbf{kk}'} (u^2_{\mathbf{k}} - v^2_{\mathbf{k}}) u_{\mathbf{k}'} v_{\mathbf{k}'} \tag{12.22}
$$

so that the total Hamiltonian is

$$
H = \sum_{\mathbf{k}} E_{\mathbf{k}} (A^{\dagger}_{\mathbf{k}} A_{\mathbf{k}} + B^{\dagger}_{\mathbf{k}} B_{\mathbf{k}}) + E_0 \tag{12.23}
$$

with

$$
E_{\mathbf{k}} = \xi_{\mathbf{k}} (u^2_{\mathbf{k}} - v^2_{\mathbf{k}}) + \frac{2}{V} \sum_{\mathbf{k}'} W_{\mathbf{kk}'} u_{\mathbf{k}'} v_{\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} \tag{12.24}
$$

and

$$
E_0 = 2 \sum_{\mathbf{k}} \xi_{\mathbf{k}} v^2_{\mathbf{k}} - \frac{1}{V} \sum_{\mathbf{kk}'} W_{\mathbf{kk}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} \tag{12.25}
$$

The operators $A^{\dagger}_{\mathbf{k}}, B^{\dagger}_{\mathbf{k}} (A_{\mathbf{k}}, B_{\mathbf{k}})$ are the creation (annihilation) operators of quasi-particles.

The eq.(12.22) can be rewritten as

$$
\xi_{\mathbf{k}} \sin 2\theta_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} \sin 2\theta_{\mathbf{k}'} W_{\mathbf{kk}'} \cos 2\theta_{\mathbf{k}} \tag{12.26}
$$

The eq.(12.24), using eq.(12.26), can be rewritten as

$$
\begin{aligned}
E_{\mathbf{k}} &= \xi_{\mathbf{k}} \cos 2\theta_{\mathbf{k}} + \xi_k \frac{(\sin 2\theta_{\mathbf{k}})^2}{\cos 2\theta_{\mathbf{k}}} \\
&= \frac{\xi_{\mathbf{k}}}{\cos 2\theta_{\mathbf{k}}}
\end{aligned} \tag{12.27}
$$

By eliminating $\xi_{\mathbf{k}}$ in eq. (12.26) using eq.(12.27), we obtain

$$E_{\mathbf{k}} \cos 2\theta_{\mathbf{k}} \sin 2\theta_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} \sin 2\theta_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \cos 2\theta_{\mathbf{k}} \tag{12.28}$$

or

$$E_{\mathbf{k}} \sin 2\theta_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} \sin 2\theta_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \tag{12.29}$$

By defining

$$\Delta_{\mathbf{k}} = E_{\mathbf{k}} \sin 2\theta_{\mathbf{k}} \tag{12.30}$$

we have

$$\Delta_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}} \tag{12.31}$$

with

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \tag{12.32}$$

Using the explicit form of $W_{\mathbf{k}\mathbf{k}'}$,

$$\Delta_{\mathbf{k}} = \frac{W}{2V} \sum_{\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}} \tag{12.33}$$

we see that Δ does not depend on $\mathbf{k}$

$$\Delta = \frac{W}{2V} \sum_{\mathbf{k}'} \frac{\Delta}{E_{\mathbf{k}'}} \tag{12.34}$$

We can now study the gap equation (12.34) which has the gapless trivial solution $\Delta = 0$. Looking for a solution with $\Delta \neq 0$, we obtain

$$1 = \frac{W}{2V} \sum_{\mathbf{k}'} \frac{1}{\sqrt{\xi_{\mathbf{k}'}^2 + \Delta^2}} \tag{12.35}$$

This equation can be studied by going into the continuum

$$1 = \frac{W}{2V} V \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{\xi(k)^2 + \Delta^2}} = \frac{W}{2} \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{\xi(k)^2 + \Delta^2}} \tag{12.36}$$

where the integral is performed around the Fermi surface $|\xi(k)| \leq \hbar\omega_D$. Notice that there is no solution for $W < 0$, case corresponding to a repulsive force.

12.2 Study of the gap equation

Let us now study the gap equation

$$\begin{aligned} 1 &= \frac{W}{2}\frac{1}{(2\pi)^3}\int_{|\xi(k)|\le\hbar\omega_D} d\Omega k^2 dk \frac{1}{\sqrt{(\epsilon(k)-\epsilon_F)^2+\Delta^2}} \\ &= \frac{W}{2}\frac{1}{(2\pi)^3}\int_{|\xi(k)|\le\hbar\omega_D} d\Omega k^2 \frac{dk}{d\epsilon}\frac{d\epsilon}{\sqrt{(\epsilon-\epsilon_F)^2+\Delta^2}} \\ &\sim \frac{W}{4}\rho_F\int_{|\xi(k)|\le\hbar\omega_D} \frac{d\epsilon}{\sqrt{(\epsilon-\epsilon_F)^2+\Delta^2}} \\ &= \frac{W}{4}\rho_F\int_{-\hbar\omega_D}^{+\hbar\omega_D}\frac{d\xi}{\sqrt{\xi^2+\Delta^2}} \\ &= \frac{W}{2}\rho_F \text{arcsinh}\frac{\hbar\omega_D}{\Delta} \end{aligned} \tag{12.37}$$

where we have introduced the density of states at the Fermi surface

$$\begin{aligned} \rho_F &= 2\frac{4\pi}{(2\pi)^3}k^2\frac{dk}{d\epsilon}|_{k_F} \\ &= \frac{1}{\pi^2}\frac{k^2}{\frac{d\epsilon}{dk}}|_{k_F} \\ &= \frac{1}{\pi^2}\frac{k_F m}{\hbar^2}|_{k_F} \end{aligned} \tag{12.38}$$

where we have introduced the Fermi momentum $\hbar k_F$.

Inverting eq.(12.37), we get the gap energy. If $W\rho_F/2 << 1$,

$$\Delta = 2\hbar\omega_D \exp{(-\frac{2}{W\rho_F})} = 2\hbar\omega_D \exp{(-\frac{2\pi^2\hbar^2}{Wmk_F})} \tag{12.39}$$

For typical metals $W\rho_F \sim 0.3-0.6$, see p.448 of ref. [10]. Then, considering $\hbar\omega_D \sim 100$ °K and $W\rho_F \sim 0.6$, we get $\Delta \sim 4$ °K.

In conclusion the energy of the first excitation is given by

$$E_{\mathbf{k}} = \sqrt{(\frac{\hbar^2\mathbf{k}^2}{2m}-\epsilon_F)^2+\Delta^2} \tag{12.40}$$

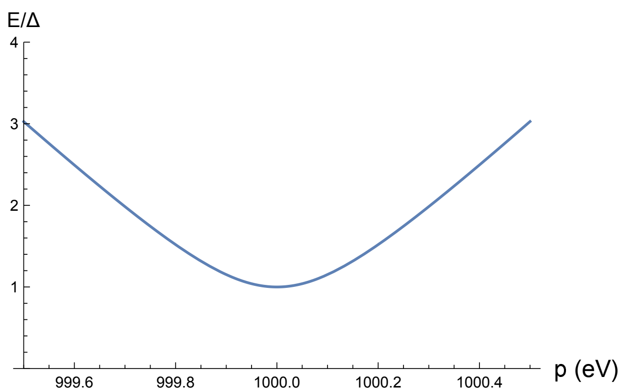

Figure 6: The energy of the first excitation, $E_{\mathbf{k}}$, rescaled by Δ, as a function of $p = |\mathbf{p}| = \hbar k$

The spectrum, as shown in Fig. 6, has a gap, meaning that one cannot create excitations with arbitrary small energy. The magnitude of this gap is Δ. The quasiparticles are mixture of electrons and holes (see eq.(12.11)). Furthermore since the quasiparticles have spin 1/2 the quasiparticles must appear in pairs, so the minimim energy is 2Δ.

12.3 Finite temperature

Let us now compute how the gap Δ depends on the temperature. Starting again from the gap equation, recall that

$$\begin{aligned}
\Delta_{\mathbf{k}} &= \frac{1}{V}\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} \\
&= \frac{1}{V}\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} < BCS|c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}',\uparrow}|BCS > \\
&= \frac{1}{V}\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} < BCS|[1 - (A^{\dagger}_{\mathbf{k}'} A_{\mathbf{k}'} + B^{\dagger}_{\mathbf{k}'} B_{\mathbf{k}'})]|BCS >
\end{aligned} \tag{12.41}$$

At $T = 0$, since there is no quasi particles, we recover eq.(12.33) and the formula for $\Delta \equiv \Delta(T = 0)$. However this method can be extended at finite temperature T. Taking the average over a statistical ensemble at temperature T we have

$$\Delta_{\mathbf{k}} = \frac{1}{V}\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} < [1-(A_{\mathbf{k}}^\dagger A_{\mathbf{k}} + B_{\mathbf{k}}^\dagger B_{\mathbf{k}})] >= \frac{1}{V}\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}'} v_{\mathbf{k}'} (1-2f(E_{\mathbf{k}'})) \tag{12.42}$$

where $f(E_{\mathbf{k}})$ is the probability to have an excitations with energy $E_{\mathbf{k}}$ at temperature T:

$$f(E_{\mathbf{k}}) = \frac{1}{1+e^{\beta E_{\mathbf{k}}}} \tag{12.43}$$

Therefore the gap equation at finite temperature becomes

$$\Delta_{\mathbf{k}} = \frac{1}{2V}\sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{E_{\mathbf{k}'}}(1-2f(E_{\mathbf{k}'})) \tag{12.44}$$

Using the explicit expression for W one gets

$$1 = \frac{W}{2V}\sum_{\mathbf{k}'} \frac{1}{E_{\mathbf{k}'}}(1-2f(E_{\mathbf{k}'})) \tag{12.45}$$

or

$$-1 + \frac{W}{2V}\sum_{\mathbf{k}'} \frac{1}{E_{\mathbf{k}'}} = \frac{W}{V}\sum_{\mathbf{k}'} \frac{f(E_{\mathbf{k}'})}{E_{\mathbf{k}'}} \tag{12.46}$$

Passing to the continuum

$$-1 + \frac{1}{2V}\frac{V}{(2\pi)^3} W \int d^3k \frac{1}{\sqrt{\xi^2(k)+\Delta(T)^2}} = \frac{1}{V}\frac{V}{(2\pi)^3} W \int d^3k f(E(k)) \frac{1}{E(k)} \tag{12.47}$$

Let us first compute the l.h. side. Proceeding as before, we get

$$\begin{aligned} l.h.side &= -1 + \frac{1}{2}W\rho_F \mathrm{asinh}\frac{\hbar\omega_D}{\Delta(T)} \\ &= -1 + \frac{1}{2}W\rho_F \ln\frac{2\hbar\omega_D}{\Delta(T)} \\ &= \frac{1}{2}W\rho_F(\ln\frac{2\hbar\omega_D}{\Delta(T)} - \frac{2}{W\rho_F}) \\ &= \frac{1}{2}W\rho_F(\ln\frac{2\hbar\omega_D}{\Delta(T)} + \ln\frac{\Delta(0)}{2\hbar\omega_D}) \\ &= \frac{1}{2}W\rho_F \ln\frac{\Delta(0)}{\Delta(T)} \end{aligned} \tag{12.48}$$

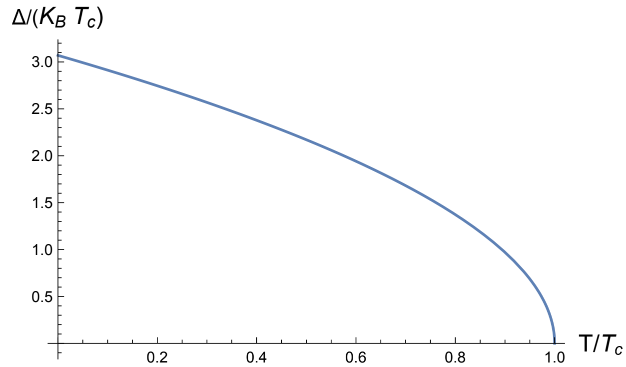

Figure 7: The gap Δ, rescaled by $K_B T_c$ as a function of T/T_c for $T < T_c$.

Summing up we have

$$\begin{aligned} \frac{1}{2} W \rho_F \ln \frac{\Delta(0)}{\Delta(T)} &= \frac{1}{2} W \rho_F \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi \frac{1}{\sqrt{\xi^2 + (\Delta(T))^2}} \frac{1}{e^{\beta\sqrt{\xi^2+(\Delta(T))^2}} + 1} \\ &\sim \frac{1}{2} W \rho_F \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{x^2+u^2}} \frac{1}{e^{\sqrt{x^2+u^2}} + 1} \end{aligned} \tag{12.49}$$

with $u = \beta\Delta(T)$, $x = \beta\xi$. The integral has been extended to $(-\infty, \infty)$ because of its rapid convergence. So:

$$\ln \frac{\Delta(0)}{\Delta(T)} = 2 \int_0^{\infty} dx \frac{1}{\sqrt{x^2+u^2}} \frac{1}{e^{\sqrt{x^2+u^2}} + 1} \tag{12.50}$$

This integral is discussed in [9]. For small Δ,

$$\ln \frac{\Delta(0)}{\Delta(T)} \sim \ln \frac{\pi k_B T}{\gamma \Delta(T)} + \frac{7\zeta(3)}{8\pi^2} \frac{(\Delta(T))^2}{(k_B T)^2} \tag{12.51}$$

where $\gamma/\pi \sim 0.57$ and $\zeta(3) \sim 1.2$. For $\Delta = 0$ we get the critical temperature:

$$k_B T_c = \frac{\gamma}{\pi} \Delta(0) \sim 0.57 \Delta(0) \tag{12.52}$$

Using eq. (12.51), expanding for small $T - T_c$, one gets that by increasing the temperature the gap becomes smaller and vanishes at T_C as

$$\Delta(T) = \sqrt{\frac{8\pi}{7\zeta(3)}} k_B T_c (1 - \frac{T}{T_c})^{\frac{1}{2}} \sim 3.07 k_B T_c (1 - \frac{T}{T_c})^{\frac{1}{2}} \tag{12.53}$$

	T_c(°K)	$\hbar\omega_D/k_B$ (°K)	$T_F \times 10^4$ (°K)	$W\rho/2$	$\Delta(T=0)/k_BT_c$
BCS					1.76
Cd	0.56	164	8.7	0.18	1.60 ± 0.05
Al	1.2	375	13.6	0.18	1.68 ± 0.05
Sn	3.75	195	11.8	0.25	1.73± 0.05
Pb	7.22	96	11.0	0.39	2.15± 0.02

Table 1: Some superconductor properties (From [10, 12, 25])

As shown in Table 12.3, the prediction of BCS theory $\Delta(T=0) \sim 1.76k_BT_c$ is quite well satisfied.

Let us finally show that $E_0 < 0$, so that $|BCS>$ is the real ground state. Using (12.27) we can write

$$\xi_{\mathbf{k}} = E_{\mathbf{k}} \cos 2\theta_{\mathbf{k}} \tag{12.54}$$

and using (12.29) we can rewrite E_0:

$$\begin{aligned} E_0 &= 2\sum_{\mathbf{k}} \xi_{\mathbf{k}} v_{\mathbf{k}}^2 - \frac{1}{V}\sum_{\mathbf{k}\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} u_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}'} v_{\mathbf{k}'} \\ &= 2\sum_{\mathbf{k}} E_{\mathbf{k}} \cos 2\theta_{\mathbf{k}} \sin^2\theta_{\mathbf{k}} - \frac{1}{2}\sum_{\mathbf{k}} E_{\mathbf{k}} \sin^2 2\theta_{\mathbf{k}} \\ &= 2\sum_{\mathbf{k}} E_{\mathbf{k}} (\cos 2\theta_{\mathbf{k}} \sin^2\theta_{\mathbf{k}} - \sin^2\theta_{\mathbf{k}} \cos^2\theta_{\mathbf{k}}) \\ &= -2\sum_{\mathbf{k}} E_{\mathbf{k}} \sin^4\theta_{\mathbf{k}} \end{aligned} \tag{12.55}$$

12.4 The BCS ground state

Let us now study the BCS ground state. It is based on the idea that electrons form Cooper pairs. The BCS vacuum is given by

$$|BCS>= \Pi_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0> \tag{12.56}$$

Thus it is a superposition of states of Cooper pairs.

This state is normalized:

$$1 =< BCS|BCS > \tag{12.57}$$

and

$$< BCS|c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}|BCS >= v_\mathbf{k}u_\mathbf{k} \tag{12.58}$$

Proof:

$$\begin{aligned} < BCS|BCS > &= < 0|\Pi_\mathbf{k}(u_\mathbf{k} + v_\mathbf{k}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow})\Pi_{\mathbf{k}'}(u_{\mathbf{k}'} + v_{\mathbf{k}'}c^\dagger_{\mathbf{k}'\uparrow}c^\dagger_{-\mathbf{k}'\downarrow})|0 > \\ &= < 0|\Pi_\mathbf{k}u_\mathbf{k}\Pi_{\mathbf{k}'}u_{\mathbf{k}'} + \Pi_\mathbf{k}v_\mathbf{k}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\Pi_{\mathbf{k}'}v_{\mathbf{k}'}c^\dagger_{\mathbf{k}'\uparrow}c^\dagger_{-\mathbf{k}'\downarrow}|0 > \\ &= \Pi_\mathbf{k}u^2_\mathbf{k}+ < 0|\Pi_\mathbf{k}\ v^2_\mathbf{k}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\Pi_{\mathbf{k}'}v_{\mathbf{k}'}c^\dagger_{\mathbf{k}'\uparrow}c^\dagger_{-\mathbf{k}'\downarrow}\delta_{\mathbf{k},\mathbf{k}'}|0 > \\ &= \Pi_\mathbf{k}(u^2_\mathbf{k} + v^2_\mathbf{k}) = 1 \end{aligned} \tag{12.59}$$

where use has been made of anticommutation relations:

$$< 0|c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}|0 >=< 0|c_{-\mathbf{k}\downarrow}(1-c^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}\uparrow})c^\dagger_{-\mathbf{k}\downarrow}|0 >=< 0|c_{-\mathbf{k}\downarrow}c^\dagger_{-\mathbf{k}\downarrow}-c_{-\mathbf{k}\downarrow}c^\dagger_{\mathbf{k}\uparrow}c_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\uparrow}|0 >= 1 \tag{12.60}$$

Furthermore

$$\begin{aligned} < BCS|c^\dagger_{\mathbf{k}\downarrow}c^\dagger_{-\mathbf{k}\uparrow}|BCS > &= < 0|\Pi_{\mathbf{k}'}(u_{\mathbf{k}'} + v_{\mathbf{k}'}c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow})c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} \\ &\quad \Pi_{\mathbf{k}''}(u_{\mathbf{k}''} + v_{\mathbf{k}''}c^\dagger_{\mathbf{k}''\uparrow}c^\dagger_{-\mathbf{k}''\downarrow})|0 > \\ &= v_\mathbf{k}u_\mathbf{k} \end{aligned} \tag{12.61}$$

The same result can be obtained by the observation that the ground state must satisfy

$$A_\mathbf{k}|BCS >= 0,\ \ B_\mathbf{k}|BCS >= 0,\ \forall\mathbf{k} \tag{12.62}$$

and therefore

$$|BCS >\sim \Pi_\mathbf{k}A_\mathbf{k}B_\mathbf{k}|0 > \tag{12.63}$$

A Conventions and units

In theoretical and experimental particle physics it is common to use *The System of Natural Units* which correspond to use

$$c = 1, \hbar = 1 \tag{A.1}$$

These two conditions reduce the three independent quantities, mass, time and length to one, usually the energy. Dimensional analysis of physical quantities are evaluated in terms of energy and all quantities are measured in eV (KeV, MeV, GeV,...).

From $c = 1$ using for example

$$E = \sqrt{p^2 + m^2} \tag{A.2}$$

we deduce that dimensions of momentum and mass are $[E]^1$, from $\hbar = 1$ and $c = 1$ since $[p][x] = [E]^0$ we get

$$[x] = [t] = [E]^{-1} \tag{A.3}$$

Useful conversion factors are listed here [18]:

$$1 \text{ eV} = 1.602176487 \; 10^{12} \text{ erg} \tag{A.4}$$

$$\frac{1 \text{ eV}}{c^2} = 1.782661758 \; 10^{-33} \text{ g} \tag{A.5}$$

$$\hbar c = 197.3269631 \text{ MeV fm} \tag{A.6}$$

where 1 fm (fermi)= 10^{-13} cm.

$$(\hbar c)^2 = 0.389379338 \text{ GeV}^2 \text{ mbarn} \tag{A.7}$$

Let us now discuss the dimensions of the fields. Since the action has the same dimension of $\hbar$, in natural units the action is dimensionless. As a consequence since

$$S = \int d^4x \mathcal{L} \tag{A.8}$$

the Lagrangian has dimensions $[E]^4$. By looking at the corresponding lagrangians, it is easy to find that the dimensions of the Klein-Gordon and electromagnetic fields are $[E]^1$ while the ones of the Dirac field are $[E]^{3/2}$.

In these lectures we use the Heaviside-Lorentz system of electromagnetic units [6], which corresponds to choose $\epsilon_0 = 1$ and $\mu_0 = 1$. In this system the Hamiltonian in the vacuum is

$$H = \frac{1}{2} \int d^3x (E^2 + B^2) \tag{A.9}$$

The non homogeneous Maxwell equations are

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} \tag{A.10}$$

The structure constant is

$$\alpha = \frac{e^2}{4\pi\hbar c} \tag{A.11}$$

or in natural units simply

$$\alpha = \frac{e^2}{4\pi} \tag{A.12}$$

B Fourier transform of the Heaviside distributions 1, δ and θ.

In this section we review the definition of Fourier transform of a temperate distribution [22]. For any distribution temperate T, defined by the linear and continuous functional

$$T : \varphi \to (T, \varphi) \tag{B.1}$$

where φ belong to the Schwartz space, the Fourier transform FT is defined by

$$(FT, \varphi) = (T, F\varphi) \tag{B.2}$$

Fourier transform of the distribution 1. For every test function φ we have

$$(F1, \varphi) = (1, F\varphi) = \int F\varphi(p)dp = (2\pi)^{n/2}\varphi(0) = ((2\pi)^{n/2}\delta, \varphi) \;\; \forall \varphi \in \mathcal{S}$$

or

$$F1 = (2\pi)^{n/2}\delta \tag{B.3}$$

Fourier transform of the distribution δ.

$$(F\delta, \varphi) = (\delta, F\varphi) = F\varphi(0) = \frac{1}{(2\pi)^{n/2}} \int \varphi(x)dx = (\frac{1}{(2\pi)^{n/2}}, \varphi) \tag{B.4}$$

therefore

$$F\delta = \frac{1}{(2\pi)^{n/2}} \tag{B.5}$$

Fourier transform of the distribution θ. Let us compute the Fourier transform of the Heaviside distribution θ:

$$F\theta(p) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \theta(x)e^{-ipx}dx = \frac{1}{i\sqrt{2\pi}} \frac{1}{p - i0} \tag{B.6}$$

Using the definition of Fourier transform of a temperate distribution, we have

$$\begin{aligned}
(F\theta, \varphi) &= (\theta, F\varphi) = \int_0^{+\infty} (F\varphi)(p)dp \\
&= \lim_{\epsilon\to 0^+} \int_0^{+\infty} e^{-\epsilon p}(F\varphi)(p)dp \\
&= \lim_{\epsilon\to 0^+} \int_0^{+\infty} e^{-\epsilon p}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi(x)e^{-ixp}dxdp \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi(x)\int_0^{+\infty} e^{-(\epsilon+ix)p}dpdx \\
&= \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi(x)\frac{1}{\epsilon+ix}dx \\
&= \frac{1}{i\sqrt{2\pi}}\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty} \frac{1}{x-i\epsilon}\varphi(x)dx \qquad \text{(B.7)}
\end{aligned}$$

where use has been made of the Fubini theorem.

We have also

$$F\theta = \frac{1}{i\sqrt{2\pi}}\ \frac{1}{p-i0} = -\frac{i}{\sqrt{2\pi}}Pv\frac{1}{p}\ +\ \sqrt{\frac{\pi}{2}}\ \delta(p). \qquad \text{(B.8)}$$

where we have made use of

$$\frac{1}{p-i0} = Pv\frac{1}{p} + i\pi\delta(p) \qquad \text{(B.9)}$$

C The distribution $\frac{1}{p-i0}$

Let us considered the tempered distribution $\log(x+iy)$. We have

$$\frac{d}{dx}\log(x+iy) = \frac{1}{x+iy} \tag{C.1}$$

Therefore

$$\lim_{y\to 0^+}\frac{1}{x+iy} = \lim_{y\to 0^+}\frac{d}{dx}\log(x+iy) \tag{C.2}$$

On the other hand we have

$$\begin{aligned}\lim_{y\to 0^+}\log(x+iy) &= \lim_{y\to 0^+}[\log|x+iy| + iArg(x+iy)] \\ &= \log|x| + i\pi\theta(-x)\end{aligned} \tag{C.3}$$

By considering the corresponding distributions se have

$$\lim_{y\to 0^+}\int \log(x+iy)\varphi(x)dx = \int[\log|x| + i\pi\theta(-x)]\,\varphi(x)dx \tag{C.4}$$

Therefore using (C.3) we obtain

$$\begin{aligned}\frac{1}{x+i0} &= \lim_{y\to 0^+}\frac{1}{x+iy} = \lim_{y\to 0^+}\frac{d}{dx}\log(x+iy) \\ &= \frac{d}{dx}\lim_{y\to 0^+}\log(x+iy) = \frac{d}{dx}[\log|x| + i\pi\theta(-x)] \\ &= Pv\frac{1}{x} - i\pi\delta(x)\end{aligned} \tag{C.5}$$

In analogous way we have

$$\frac{1}{x-i0} = Pv\frac{1}{x} + i\pi\delta(x) \tag{C.6}$$

D Coherent states

Aim of this Appendix is the definition and the study of the properties of the coherent states. The coherent states $|c>$ are defined as eigenvectors of the annihilation operator of the harmonic oscillator a. Their explicit from is given by

$$|c>= A^{1/2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}|n> \tag{D.1}$$

where

$$|n>= \frac{(a^\dagger)^n}{\sqrt{n!}}|0> \tag{D.2}$$

and

$$A = \exp\left(-|c|^2\right) \tag{D.3}$$

c is a complex number since the operator a is not hermitian. These states satisfy the following properties:

$$< c|c >= 1 \tag{D.4}$$

$$|c >= A^{1/2}\exp\left(ca^\dagger\right)|0 > \tag{D.5}$$

$$a|c >= c|c > \tag{D.6}$$

$$< c|N|c >=< c|a^\dagger a|c >= |c|^2 \tag{D.7}$$

In fact we have

$$\begin{aligned}< c|c > &= A\sum_n\sum_{n'} < n|n' > \frac{1}{\sqrt{n!}\sqrt{n'!}}(c^*)^n c^n \\ &= \exp\left(-|c|^2\right)\exp\left(|c|^2\right) = 1 \end{aligned} \tag{D.8}$$

$$|c >= A^{1/2}\sum_{n=0}^{\infty}\frac{c^n}{\sqrt{n!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|0 >= A^{1/2}\exp\left(ca^\dagger\right)|0 > \tag{D.9}$$

$$a|c >= \sum_{n=1}^{\infty}\frac{c^n}{\sqrt{(n-1)!}}|n-1 >= c|c > \tag{D.10}$$

Finally eq.(D.7) derives directly from eq.(D.6).

As an application let us build the coherent states for the electromagnetic field using the creation operator of a photon with momentum $\mathbf{k}$ and polarization α

$$|c^\alpha_{\mathbf{k}}>= A^{1/2}\sum_{n=0}^{\infty}\frac{(c^\alpha_{\mathbf{k}})^n}{\sqrt{n!}}|n> \tag{D.11}$$

with

$$|n>=\frac{(a^{\alpha\dagger}_{\mathbf{k}})^n}{\sqrt{n!}}|0> \tag{D.12}$$

We can then compute the expectation value of the electric field in the radiation gauge, $\mathbf{E}(t,\mathbf{x}) = -\dot{\mathbf{A}}(t,\mathbf{x})$, which turns out to be equal to

$$\begin{aligned}
< c^\alpha_{\mathbf{k}}|\mathbf{E}(t,\mathbf{x})|c^\alpha_{\mathbf{k}} > &= -< c^\alpha_{\mathbf{k}}|\sum_{\mathbf{k}'}\sum_{\alpha'=1,2}\frac{1}{\sqrt{2V\omega_{k'}}}(-i\omega_{k'})\boldsymbol{\epsilon}^{\alpha'}_{\mathbf{k}'}[a^{\alpha'}_{\mathbf{k}'}e^{-ik'x} - h.c.]|c^\alpha_{\mathbf{k}} > \\
&= i\epsilon^\alpha_{\mathbf{k}}\sqrt{\frac{\omega_k}{2V}}[c^\alpha_{\mathbf{k}}e^{-ikx} - c.c.] \\
&= -\epsilon^\alpha_{\mathbf{k}}\sqrt{\frac{2\omega_k}{V}}|c^\alpha_{\mathbf{k}}|\sin\left(\mathbf{k}\cdot\mathbf{x} - \omega_k t + \delta^\alpha_{\mathbf{k}}\right)
\end{aligned} \tag{D.13}$$

where

$$c^\alpha_{\mathbf{k}} = |c^\alpha_{\mathbf{k}}|\exp\left(i\delta^\alpha_{\mathbf{k}}\right) \tag{D.14}$$

If we instead consider the expectation value of the electric field $\mathbf{E}(t,\mathbf{x})$ on a state with a definite number of photons $|N^{\alpha_1}_{k_1}>$, the result is zero

$$< N^{\alpha_1}_{k_1}|\mathbf{E}(t,\mathbf{x})|N^{\alpha_1}_{k_1} >= 0 \tag{D.15}$$

In conclusion the coherent state is the quantum state which is closer to a classical electromagnetic wave.

E Path integral for field theory

Path integral for field theory is constructed by using the same procedure of path integral for quantum mechanics. It is convenient, since the quantum fields are expressed in terms of creation and annihilation operators, to consider the eigenvectors of such operators, the coherent states. Let us consider a set of creation operators $a_i^\dagger$ and build the coherent state

$$|\phi >= \exp{(\sum_i \phi_i a_i^\dagger)}|0 > \tag{E.1}$$

where ϕ_i are a set of complex numbers. As we have seen in the Appendix D we have

$$a_i|\phi >= \phi_i|\phi > \tag{E.2}$$

Notice that, taking the hermitian conjugate, we obtain

$$< \phi|a_i^\dagger =< \phi|\bar{\phi}_i \tag{E.3}$$

where $\bar{\phi}_i$ denotes the complex conjugate of ϕ_i, and

$$< \eta|\phi >= \exp{(\sum_i \bar{\eta}_i \phi_i)} \tag{E.4}$$

These states are not normalized

$$< \phi|\phi >= \exp{(\sum_i \bar{\phi}_i \phi_i)} \tag{E.5}$$

We have

$$\begin{aligned} < \phi|\phi > &= < 0| \exp{(\sum_j \bar{\phi}_j a_j)} \exp{(\sum_i \phi_i a_i^\dagger)}|0 > \\ &= < 0| \exp{(\sum_i \phi_i a_i^\dagger)} \exp{(\sum_j \bar{\phi}_j a_j)}|0 > \exp{(\sum_i \bar{\phi}_i \phi_i)} \\ &= \exp{(\sum_i \bar{\phi}_i \phi_i)} \end{aligned} \tag{E.6}$$

where we have used

$$\exp{(A)}\exp{(B)} = \exp{(A+B)}\exp{(\frac{1}{2}[A,B])} = \exp{(\frac{1}{2}[A,B])}\exp{(B)}\exp{(A)} \tag{E.7}$$

which holds provided the commutator be a c-number.

The coherent states satisfy also a completeness relation

$$\int \Pi_i \frac{1}{\pi} d\bar{\phi}_i d\phi_i \exp{(-\sum_i \bar{\phi}_i \phi_i)} |\phi >< \phi| = I \tag{E.8}$$

where $d\bar{\phi}_i d\phi_i = d\, Re\, \phi_i d\, Im\, \phi_i$. The proof of (E.8) is based on Schur's Lemma[32]. First we note that $a_i, a_i^\dagger$ act irreducibly on the Fock space.Then we need to show that the left hand side of eq. (E.8) commute with a_i and $a_i^\dagger$. We have

$$\begin{aligned} a_i \int d\bar{\phi} d\phi \exp{(-\sum_i \bar{\phi}_i \phi_i)} |\phi >< \phi| &= \int d\bar{\phi} d\phi \ \exp{(-\sum_i \bar{\phi}_i \phi_i)} \phi_i |\phi >< \phi| \\ &= -\int d\bar{\phi} d\phi \frac{\partial}{\partial \bar{\phi}_i} [\exp{(-\sum_i \bar{\phi}_i \phi_i)}] |\phi >< \phi| \\ &= \int d\bar{\phi} d\phi [\exp{(-\sum_i \bar{\phi}_i \phi_i)}] |\phi > \frac{\partial}{\partial \bar{\phi}_i} < \phi| \\ &= \int d\bar{\phi} d\phi \exp{(-\sum_i \bar{\phi}_i \phi_i)} |\phi >< \phi| a_i \end{aligned} \tag{E.9}$$

where we have set $d\bar{\phi} d\phi \equiv = \Pi_i \frac{1}{\pi} d\bar{\phi}_i d\phi_i$. Taking the adjoint one can also check that also $a_i^\dagger$ commute with the left hand side of eq. (E.8). The Schur's Lemma then guarantees that the left hand side of eq. (E.8) is multiple of the identity operator.

The normalization is chosen so that

$$\int d\bar{\phi} d\phi \exp{(-\sum_i \bar{\phi}_i \phi_i)} < 0|\phi >< \phi|0 >= \int d\bar{\phi} d\phi \exp{(-\sum_i \bar{\phi}_i \phi_i)} = 1 \tag{E.10}$$

Notice that we have also

$$a_i^\dagger |\phi >= \frac{\partial}{\partial \phi_i} |\phi > \tag{E.11}$$

[32]Schur Lemma. Let $S(G)$ be an irreducible representation of a group G on the vector space V and A an operator on V. If $[A, S(g)] = 0\ \forall g \in G$ then A is multiple of the identity operator, $A = \lambda I$. For the proof, see [24]

Let us now consider the partition function:

$$Z = Tr \exp[-\beta H] = \sum_n < n|\exp[-\beta H]|n >= \tag{E.12}$$

where $\beta = 1/k_B T$. Using eq.(E.8) we can pass to the coherent state representation

$$\begin{aligned} Z &= \int d\bar{\phi} d\phi \exp\left(-\sum_i \bar{\phi}_i \phi_i\right) \sum_n < n|\phi >< \phi|\exp[-\beta H]|n > \\ &= \int d\bar{\phi} d\phi \exp\left(-\sum_i \bar{\phi}_i \phi_i\right) \sum_n < \phi|\exp[-\beta H]|n >< n|\phi > \\ &= \int d\bar{\phi} d\phi \exp\left(-\sum_i \bar{\phi}_i \phi_i\right) < \phi|\exp[-\beta H]|\phi > \end{aligned} \tag{E.13}$$

Notice that in order to use the completeness relation

$$\sum_n |n >< n| = 1 \tag{E.14}$$

we have commuted $< n|\phi >$ with $< \phi|n >$. In the case of fermions this (anti) commutation gives a minus sign. We can now repeat the derivation of path integral. Let us assume the following Hamiltonian

$$H = \sum_{ij} k_{ij} a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l \tag{E.15}$$

and divide the time interval β in N interval of length δ. Then we have ($\phi \equiv \phi_i$ and the sum over i is understood)

$$\begin{aligned} Z &= \int d\bar{\phi} d\phi \exp\left(-\sum_i \bar{\phi}_i \phi_i\right) < \phi|\exp[-\beta H]|\phi > \\ &= \lim_{\delta\to 0} \int \Pi_{n=0}^{N-1} d\bar{\phi}^n d\phi^n \exp\left(-\bar{\phi}^n \phi^n\right) < \phi^0|\exp[-\delta H]|\phi^1 > \ldots < \phi^{N-1}|\exp[-\delta H]|\phi^N > \end{aligned} \tag{E.16}$$

where $\phi^0 = \phi^N = \phi$.

Now

$$\begin{aligned} < \phi^{n+1}|\exp[-\beta H]|\phi^n > &\sim < \phi^{n+1}|\phi^n > -\delta < \phi^{n+1}|H|\phi^n > \\ &= < \phi^{n+1}|\phi^n > (1 - \delta \frac{< \phi^{n+1}|H|\phi^n >}{< \phi^{n+1}|\phi^n >}) \\ &= < \phi^{n+1}|\phi^n > (1 - \delta H(\bar{\phi}^{n+1}, \phi^n)) \\ &= \exp(\bar{\phi}^{n+1}\phi^n)(1 - \delta H(\bar{\phi}^{n+1}, \phi^n)) \\ &\sim \exp(\bar{\phi}^{n+1}\phi^n)\exp[-\delta H(\bar{\phi}^{n+1}, \phi^n)] \quad \text{(E.17)} \end{aligned}$$

where $H(\bar{\phi}^{n+1}, \phi^n)$ is the function obtained by the substitution $a^\dagger \to \bar{\phi}$, $a \to \phi$.

By substituting eq.(E.17) in eq.(E.16), we obtain

$$\begin{aligned} Z &= \lim_{\delta \to 0} \int \Pi_{n=0}^{N-1} d\bar{\phi}^n d\phi^n \exp[((\bar{\phi}^{n+1} - \bar{\phi}^n)]\phi^n] \exp[-\delta H(\bar{\phi}^{n+1}, \phi^n)] \\ &= \lim_{\delta \to 0} \int \Pi_{n=0}^{N-1} d\bar{\phi}^n d\phi^n \exp + [\delta((\frac{(\bar{\phi}^{n+1} - \bar{\phi}^n)\phi^n}{\delta} - H(\bar{\phi}^{n+1}, \phi^n)] \\ &\equiv \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \exp[-\int_0^\beta d\tau L] \quad \text{(E.18)} \end{aligned}$$

where

$$\begin{aligned} L &= -\dot{\bar{\phi}}\phi + H(\bar{\phi}, \phi) \\ &= \sum_{i,j}[-\dot{\bar{\phi}}_i\phi_j + \sum_{i,j} k_{ij}\bar{\phi}_i\phi_j + \sum_{ijkl} V_{ijkl}\bar{\phi}_i\bar{\phi}_j\phi_k\phi_l] \quad \text{(E.19)} \end{aligned}$$

By going back to the configuration space we obtain

$$Z = \int \mathcal{D}\bar{\phi}(\tau, x)\mathcal{D}\bar{\phi}(\tau, x)e^{-S} \quad \text{(E.20)}$$

where

$$S = \int_0^\beta d\tau \int d^3x[\bar{\phi}\partial_\tau\phi + \frac{1}{2m}(\nabla\phi)^*(\nabla\phi) + V(x-y)\bar{\phi}(x)\phi(x)\bar{\phi}(y)\phi(y)] \quad \text{(E.21)}$$

The functional integration is performed over all the fields satisfying periodic boundary conditions

$$\phi(\tau, x) = \phi(\tau + \beta, x) \quad \text{(E.22)}$$

which imply

$$\phi(\tau, x) = \sum_{n=-\infty}^{\infty} e^{i\omega_n \tau} \phi(\omega_n, x) \tag{E.23}$$

where

$$\omega_n = \frac{2\pi n}{\beta} \tag{E.24}$$

As we have already noticed for fermions, we require antiperiodic boundary conditions

$$\psi(\tau, x) = -\psi(\tau + \beta, x) \tag{E.25}$$

which imply

$$\phi(\tau, x) = \sum_{n=-\infty}^{\infty} e^{i\omega_n \tau} \phi(\omega_n, x) \tag{E.26}$$

where

$$\omega_n = \frac{\pi(2n+1)}{\beta} \tag{E.27}$$

F Yukawa potential

Let us consider the static equation for the Klein-Gordon field in presence of a point-like source

$$(-\nabla^2 + m^2)E(\mathbf{x}) = \delta^3(x) \tag{F.1}$$

By Fourier transforming, we obtain

$$(\mathbf{q}^2 + m^2)FE(q) = (2\pi)^{-3/2} \tag{F.2}$$

where $FE(q)$ denotes the Fourier transform of E. Therefore

$$FE(q) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2 + m^2} \tag{F.3}$$

where $q = |\mathbf{q}|$, and

$$E(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3q FE(q) \exp(i\mathbf{q}\cdot\mathbf{x}) \tag{F.4}$$

Therefore

$$\begin{aligned} E(\mathbf{x}) &= \frac{1}{(2\pi)^3} \int d^3q \frac{1}{q^2+m^2} \exp(i\mathbf{q}\cdot\mathbf{x}) \\ &= \frac{1}{(2\pi)^3} \int_0^\infty dq \frac{q^2}{q^2+m^2} \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi \exp(iqx\cos\theta) \\ &= \frac{1}{2\pi^2} \int_0^\infty dq \frac{q^2}{q^2+m^2} \frac{\sin qx}{qx} \\ &= \frac{1}{2\pi^2} \int_{-\infty}^\infty dq \frac{q}{q^2+m^2} \frac{\sin qx}{x} \end{aligned} \tag{F.5}$$

where we recall $x = |\mathbf{x}|$. The integral can be compute in the complex plane by using $\sin qx = (2i)^{-1}(e^{iqx} - e^{-iqx})$ and closing the contour above for the first exponential or below for the second. By using the residue theorem we get

$$E(\mathbf{x}) = \frac{e^{-m|\mathbf{x}|}}{4\pi|\mathbf{x}|} \tag{F.6}$$

In conclusion we get a screened Coulomb potential with a range of order m^{-1}. For $m = 0$ we recover, apart the sign, the Coulomb potential.

G Dirac equation solutions and their properties

G.1 Spinors

Solutions of the Dirac equations can be written, using the solutions in the rest frame of the electrons, as

$$u_r(\mathbf{p}) = \frac{\hat{p}+m}{\sqrt{2m(E+m)}}u_r(\mathbf{0}), \ \ v_r(\mathbf{p}) = \frac{-\hat{p}+m}{\sqrt{2m(E+m)}}v_r(\mathbf{0}), \ \ r=1,2 \quad \text{(G.1)}$$

i) the solutions are normalized

$$\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = \delta_{rs}, \ \ \bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = -\delta_{rs} \quad \text{(G.2)}$$

In fact we have

$$\begin{aligned}
\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) &= \frac{1}{2m(m+E)}u^\dagger(\mathbf{0})_r(\hat{p}^\dagger+m)\gamma^0(\hat{p}+m)u_s(\mathbf{0}) \\
&= \frac{1}{2m(m+E)}\bar{u}(\mathbf{0})\gamma^0(\hat{p}^\dagger+m)\gamma^0(\hat{p}+m)u_s(\mathbf{0}) \\
&= \frac{1}{2m(m+E)}\bar{u}(\mathbf{0})(\hat{p}+m)^2u_s(\mathbf{0}) \\
&= \frac{1}{(m+E)}\bar{u}_r(\mathbf{0})(\hat{p}+m)u_s(\mathbf{0}) \\
&= \bar{u}_r(\mathbf{0})u_s(\mathbf{0}) \\
&= \delta_{rs} \qquad \text{(G.3)}
\end{aligned}$$

where we have used

$$\bar{u}_r(\mathbf{0})\gamma^i u_s(\mathbf{0}) = 0 \quad \text{(G.4)}$$

We have in fact

$$\bar{u}_r(\mathbf{0})\gamma^i u_s(\mathbf{0}) = u_r(\mathbf{0})^\dagger\gamma^i u_s(\mathbf{0}) = u_r^\dagger(\mathbf{0})\gamma^i\gamma^0 u_s(\mathbf{0}) = -\bar{u}_r(\mathbf{0})\gamma^i u_s(\mathbf{0}) = 0 \quad \text{(G.5)}$$

In similar way one can prove the normalization for the v.

ii) By introducing the two-component non relativistic spinors

$$\chi_1 = \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{G.6}$$

the solutions can be written using the Dirac-Pauli representation of the Dirac matrices as

$$u_r(\mathbf{p}) = A \begin{pmatrix} \chi_r \\ B\mathbf{p}\cdot\boldsymbol{\sigma}\chi_r \end{pmatrix}, \quad v_r(\mathbf{p}) = A \begin{pmatrix} B\mathbf{p}\cdot\boldsymbol{\sigma}\eta_r \\ \eta_r \end{pmatrix}, \tag{G.7}$$

with

$$A = \left(\frac{E+m}{2m}\right)^{1/2}, \quad B = \frac{1}{E+m} \tag{G.8}$$

The explicit form is

$$u_1(\mathbf{p}) = A \begin{pmatrix} 1 \\ 0 \\ Bp_3 \\ B(p_1 + ip_2) \end{pmatrix}, \quad u_2(\mathbf{p}) = A \begin{pmatrix} 0 \\ 1 \\ B(p_1 - ip_2) \\ -Bp_3 \end{pmatrix} \tag{G.9}$$

$$v_1(\mathbf{p}) = A \begin{pmatrix} B(p_1 - ip_2) \\ -Bp_3 \\ 1 \\ 0 \end{pmatrix}, \quad v_2(\mathbf{p}) = A \begin{pmatrix} Bp_3 \\ B(p_1 + ip_2) \\ 0 \\ 1 \end{pmatrix} \tag{G.10}$$

In fact we have

$$\begin{aligned} u_r(\mathbf{p}) &= \frac{\hat{p}+m}{\sqrt{2m(E+m)}} u_r(\mathbf{0}) \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m & -\mathbf{p}\cdot\boldsymbol{\sigma} \\ \mathbf{p}\cdot\boldsymbol{\sigma} & -E+m \end{pmatrix} \begin{pmatrix} \chi_r \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} A\chi_r \\ AB\mathbf{p}\cdot\boldsymbol{\sigma}\chi_r \end{pmatrix} \end{aligned} \tag{G.11}$$

iv) The spinors satisfy

$$u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) = \delta_{rs}\frac{E}{m} = v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) \tag{G.12}$$

In fact we have

$$\begin{aligned}
u_r^\dagger(\mathbf{p})u_s(\mathbf{p}) &= \frac{1}{2m(m+E)}u_r^\dagger(\mathbf{0})(\hat{p}^\dagger+m)(\hat{p}+m)u_s(\mathbf{0}) \\
&= \frac{1}{2m(m+E)}\bar{u}_r(\mathbf{0})(\hat{p}+m)\gamma^0(\hat{p}+m)u_s(\mathbf{0}) \\
&= \frac{1}{m(m+E)}\bar{u}_r(\mathbf{0})E(m+\hat{p})u_s(\mathbf{0}) \\
&= \frac{1}{m(m+E)}\bar{u}_r(\mathbf{0})E(m+E)u_s(\mathbf{0}) \\
&= \frac{E}{m}\delta_{rs} \qquad \text{(G.13)}
\end{aligned}$$

where we have used

$$\begin{aligned}
(\hat{p}+m)\gamma^0(\hat{p}+m) &= (\hat{p}+m)(\gamma^0\gamma^\mu p_\mu+\gamma^0 m) \\
&= (\hat{p}+m)(2g^{0\mu}p_\mu-\hat{p}\gamma^0+\gamma^0 m) \\
&= (\hat{p}+m)(2E-\hat{p}\gamma^0+\gamma^0 m) \\
&= 2E(\hat{p}+m)+(\hat{p}+m)(-\hat{p}+m)\gamma^0 \\
&= 2E(\hat{p}+m) \qquad \text{(G.14)}
\end{aligned}$$

Similarly for v.

v) The spinors satisfy also

$$u^\dagger(\mathbf{p})v(-\mathbf{p}) = 0 = v^\dagger(\mathbf{p})u(-\mathbf{p}) \qquad \text{(G.15)}$$

and

$$\bar{u}(\mathbf{p})v(\mathbf{p}) = 0 \qquad \text{(G.16)}$$

In fact

$$\begin{aligned}
u^\dagger(\mathbf{p})v(-\mathbf{p}) &= u^\dagger(\mathbf{0})(\hat{p}^\dagger+m)(-E\gamma^0-p^k\gamma^k+m)v(\mathbf{0}) \\
&= \bar{u}(\mathbf{0})\gamma^0(\hat{p}^\dagger+m)(-E\gamma^0-p^k\gamma^k+m)v(\mathbf{0}) \\
&= \bar{u}(\mathbf{0})(\hat{p}+m)\gamma^0(-E\gamma^0-p^k\gamma^k+m)v(\mathbf{0}) \\
&= \bar{u}(\mathbf{0})(\hat{p}+m)(-E\gamma^0+p^k\gamma^k+m)\gamma^0 v(\mathbf{0}) \\
&= -\bar{u}(\mathbf{0})(E^2-\mathbf{p}^2-m^2)\gamma^0 v(\mathbf{0}) \\
&= 0 \qquad \text{(G.17)}
\end{aligned}$$

$$
\begin{aligned}
\bar{u}(\mathbf{p})v(\mathbf{p}) &= u^\dagger(\mathbf{0})(\hat{p}^\dagger + m)\gamma^0(-\hat{p}+m)v(\mathbf{0}) \\
&= u^\dagger(\mathbf{0})\gamma^0(\hat{p} + m)(-\hat{p}+m)v(\mathbf{0}) \\
&= -\bar{u}(\mathbf{0})(E^2 - \mathbf{p}^2 - m^2)v(\mathbf{0}) \\
&= 0 \qquad \text{(G.18)}
\end{aligned}
$$

Solutions of the Dirac equation can also be built using the helicity operator defined as

$$
h(\mathbf{p}) = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{|\mathbf{p}|} \qquad \text{(G.19)}
$$

where σ is the 4 by 4 matrix

$$
\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \qquad \text{(G.20)}
$$

This operator, which commutes with the Hamiltonian $H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m$, corresponds to the projection of twice the spin of the particle in the direction of motion. Since

$$
h(\mathbf{p})^2 = 1 \qquad \text{(G.21)}
$$

the eigenvalues of the elicity operator are ± 1. Solutions of the Dirac equation in terms of helicity spinors can be found in [5].

G.2 Projection operators

The energy projection operators are

$$
\Lambda^{\pm}(\mathbf{p}) = \frac{\pm\hat{p} + m}{2m} \qquad \text{(G.22)}
$$

They satisfy

$$
(\Lambda^{\pm}(\mathbf{p}))^2 = \Lambda^{\pm}(\mathbf{p}), \;\; \Lambda^{+}(\mathbf{p}) + \Lambda^{-}(\mathbf{p}) = 1, \;\; \Lambda^{+}(\mathbf{p})\Lambda^{-}(\mathbf{p}) = 0 \qquad \text{(G.23)}
$$

They project out positive and negative energy states

$$
\Lambda^{+}(\mathbf{p})u_r(\mathbf{p}) = u_r(\mathbf{p}), \;\; \Lambda^{-}(\mathbf{p})v_r(\mathbf{p}) = v_r(\mathbf{p}) \qquad \text{(G.24)}
$$

One can easily show that

$$\Lambda^+_{\alpha\beta}(\mathbf{p}) = \sum_r u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}), \tag{G.25}$$

In fact, in the rest frame, we have

$$\begin{aligned}
\sum_r u_r(\mathbf{0})\bar{u}_r(\mathbf{0}) &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1+\gamma^0}{2}
\end{aligned} \tag{G.26}$$

Therefore

$$\begin{aligned}
\Lambda^+ &= \sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{1}{2m(m+E)}(\hat{p}+m)\sum_r u_r(\mathbf{0})u_r^\dagger(\mathbf{0})(\hat{p}^\dagger+m)\gamma^0 \\
&= \frac{1}{2m(m+E)}(\hat{p}+m)\sum_r u_r(\mathbf{0})u_r^\dagger(\mathbf{0})\gamma^0\gamma^0(\hat{p}^\dagger+m)\gamma^0 \\
&= \frac{1}{2m(m+E)}(\hat{p}+m)\frac{1+\gamma^0}{2}(\hat{p}+m) \\
&= \frac{1}{4m(m+E)}[(\hat{p}+m)^2 + (\hat{p}+m)\gamma^0(\hat{p}+m)] \\
&= \frac{1}{2m(m+E)}(m^2 + \hat{p}m + E(\hat{p}+m)) \\
&= \frac{\hat{p}+m}{2m}
\end{aligned} \tag{G.27}$$

where we have used (G.14). In analogous way

$$\Lambda^-_{\alpha\beta}(\mathbf{p}) = -\sum_r v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p}) = \frac{-\hat{p}+m}{2m} \tag{G.28}$$

One can build also spin projectors. In the rest frame they are simply

$$P_S^{\pm} = \frac{1 \pm \sigma_{12}}{2} = \frac{1}{2}\begin{pmatrix} 1 \pm \sigma_3 & 0 \\ 0 & 1 \pm \sigma_3 \end{pmatrix} \tag{G.29}$$

with σ_{12} given by eq.(10.58). They project spin $\pm 1/2$ solutions in the 3d direction. In particular P_S^+ projects $u_1(\mathbf{0}), v_1(\mathbf{0})$ while P_S^- projects $u_2(\mathbf{0}), v_2(\mathbf{0})$. In a general frame one prefers to consider

$$\tilde{P}_S^{\pm} = \frac{1 \pm \gamma_5 \hat{n}}{2} \tag{G.30}$$

where n^μ is a space like four vector orthogonal to p^μ

$$n^2 = -1, \quad n^\mu p_\mu = 0 \tag{G.31}$$

Now, in the rest frame

$$\tilde{P}_S = \frac{1 \pm \sigma_{12}\gamma_0}{2} = \frac{1}{2}\begin{pmatrix} 1 \pm \sigma_3 & 0 \\ 0 & 1 \mp \sigma_3 \end{pmatrix} \tag{G.32}$$

and therefore $\tilde{P}_S^+$ projects $u_1(\mathbf{0}), v_2(\mathbf{0})$ while $\tilde{P}_S^-$ projects $u_2(\mathbf{0}), v_1(\mathbf{0})$.

In the rest frame one has $n^0 = 0$ and therefore one can always choose $\mathbf{n} = (0, 0, 1)$.

G.3 Trace theorems

$$\text{i) } \operatorname{tr}\gamma^\mu = 0, \;\; \text{ii) } \operatorname{tr}\gamma_5 = 0, \;\; \text{iii) } \operatorname{tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu} \tag{G.33}$$

$$\text{iv) } \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0, \;\; \text{v) } \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{G.34}$$

$$\text{vi) } \operatorname{tr}\gamma_5\gamma^\mu\gamma^\nu = 0 \;\; \text{vii) } \operatorname{tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4i\epsilon^{\mu\nu\rho\sigma} \tag{G.35}$$

In general the trace of an odd numbers of γ's is zero.

Proofs:

i)

$$\operatorname{tr}\gamma^i = \operatorname{tr}(\gamma^0)^2\gamma^i = \operatorname{tr}\gamma^0\gamma^i\gamma^0 = -\operatorname{tr}(\gamma^0)^2\gamma^i = -\operatorname{tr}\gamma^i = 0 \tag{G.36}$$

and

$$\mathrm{tr}\,\gamma^0 = \mathrm{tr}\,(\gamma^i)^2\gamma^0 = \mathrm{tr}\,\gamma^i\gamma^0\gamma^i = -\mathrm{tr}\,(\gamma^i)^2\gamma^0 = -\mathrm{tr}\,\gamma^0 = 0 \tag{G.37}$$

ii)

$$\mathrm{tr}\,\gamma_5 = i\mathrm{tr}\,(\gamma^0)^2\gamma_5 = i\mathrm{tr}\,\gamma^0\gamma_5\gamma^0 = -i\mathrm{tr}\,(\gamma^0)^2\gamma_5 = \mathrm{tr}\,\gamma_5 = 0 \tag{G.38}$$

iii)

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu = \frac{1}{2}\mathrm{tr}\,(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \frac{1}{2}2g^{\mu\nu}\mathrm{tr}\,I_4 = 4g^{\mu\nu} \tag{G.39}$$

iv)

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho = \mathrm{tr}\,\gamma_5^2\gamma^\mu\gamma^\nu\gamma^\rho = \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma_5 = -\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho \tag{G.40}$$

v)

$$\begin{aligned}\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= 2g^{\mu\nu}\mathrm{tr}\,\gamma^\rho\gamma^\sigma - \mathrm{tr}\,\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma \\ &= 8g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}\mathrm{tr}\,\gamma^\nu\gamma^\sigma + \mathrm{tr}\,\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma \\ &= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 2g^{\mu\sigma}\mathrm{tr}\,\gamma^\nu\gamma^\rho - \mathrm{tr}\,\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu \\ &= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\sigma}g^{\nu\rho} - \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\end{aligned} \tag{G.41}$$

or

$$\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{G.42}$$

vi) When $\mu = \nu$ vi) is equivalent to ii). For $\mu \neq \nu$, using

$$\gamma_5 = \frac{i}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta \tag{G.43}$$

we can show that

$$\gamma^\mu\gamma^\nu = -i\gamma_5\epsilon^{\mu\nu\rho\sigma}\gamma_\rho\gamma_\sigma \ \text{ no sum on } \rho,\sigma \tag{G.44}$$

Then

$$\mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu = -i\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,(\gamma_5)^2\gamma_\rho\gamma_\sigma = -i\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\,\gamma_\rho\gamma_\sigma = 0 \tag{G.45}$$

where we have used $\mathrm{tr}\,\gamma_\rho\gamma_\sigma = 0$ since $\rho \neq \sigma$.

vii) When $\mu = \nu = \rho = \sigma$ vii) is equivalent to ii); when two indices are equal vii) is equivalent to vi). Let us consider all four indices different. Then

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\epsilon^{\mu\nu\rho\sigma}\gamma_5 \tag{G.46}$$

and

$$\mathrm{tr}\,\gamma_5 = \mathrm{tr}\,\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\,\gamma_5^2 = 4i\epsilon^{\mu\nu\rho\sigma} \tag{G.47}$$

H Calculation of $\mu \to e\bar{\nu}_e\nu_\mu$ decay squared amplitude

Aim of this Appendix is the calculation of the sum over the final spins and the average of the initial spin of the squared amplitude

$$\frac{1}{2}\sum_{r_i,r_f} |\mathcal{M}_{fi}|^2 \tag{H.1}$$

where

$$\mathcal{M}_{fi} = \bar{u}_{r_{\nu_e}}(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e})\bar{u}_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)u_{r_\mu}(\mathbf{p}_\mu) \tag{H.2}$$

We have

$$\begin{aligned}
\frac{1}{2}\sum_{r_i,r_f} |\mathcal{M}_{fi}|^2 &= \frac{1}{2}\sum_{r_i,r_f} \bar{u}_{r_e}(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e})\bar{u}_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)u_{r_\mu}(\mathbf{p}_\mu) \\
&\quad u^\dagger_{r_\mu}(\mathbf{p}_\mu)(1-\gamma_5)\gamma^\dagger_\sigma\gamma_0 u_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e})^\dagger(1-\gamma_5)\gamma^{\sigma\dagger}\gamma_0 u_{r_e}(\mathbf{p}_e) \\
&= \frac{1}{2}\sum_{r_i,r_f} \bar{v}_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e})\gamma_0(1-\gamma_5)\gamma^{\sigma\dagger}\gamma_0 u_{r_e}(\mathbf{p}_e)\bar{u}_{r_e}(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)v_{r_{\bar{\nu}_e}}(\mathbf{p}_{\bar{\nu}_e}) \\
&\quad \bar{u}_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)u_{r_\mu}(\mathbf{p}_\mu)\bar{u}_{r_\mu}(\mathbf{p}_\mu)\gamma_0(1-\gamma_5)\gamma^\dagger_\sigma\gamma_0 u_{r_{\nu_\mu}}(\mathbf{p}_{\nu_\mu}) \\
&= \frac{1}{2}Tr[(-\Lambda^-(\mathbf{p}_{\bar{\nu}_e}))\gamma_0(1-\gamma_5)\gamma^{\sigma\dagger}\gamma_0\Lambda^+(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)] \\
&\quad Tr[\Lambda^+(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)\Lambda^+(\mathbf{p}_\mu)\gamma_0(1-\gamma_5)\gamma^\dagger_\sigma\gamma_0 \\
&= \frac{1}{2}Tr[(-\Lambda^-(\mathbf{p}_{\bar{\nu}_e}))\gamma^\sigma(1-\gamma_5)\Lambda^+(\mathbf{p}_e)\gamma_\lambda(1-\gamma_5)] \\
&\quad Tr[\Lambda^+(\mathbf{p}_{\nu_\mu})\gamma^\lambda(1-\gamma_5)\Lambda^+(\mathbf{p}_\mu)\gamma_\sigma(1-\gamma_5)] \\
&= \frac{1}{2}\frac{1}{16 m_\mu m_{\nu_e} m_{\nu_\mu} m_e} Tr[\hat{p}_{\nu_e}\gamma_\sigma(1-\gamma_5)\hat{p}_e\gamma^\lambda(1-\gamma_5)] \\
&\quad Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)(\hat{p}_\mu + m_\mu)\gamma^\sigma(1-\gamma_5)]
\end{aligned} \tag{H.3}$$

where in the projection operators Λ we have neglected the masses of th electron and neutrinos with respect to the μ mass.

Now

$$Tr[\hat{p}_{\nu_e}\gamma_\sigma(1-\gamma_5)\hat{p}_e\gamma^\lambda(1-\gamma_5)] = 2Tr[\hat{p}_{\nu_e}\gamma_\sigma\hat{p}_e\gamma^\lambda] - 2Tr[\hat{p}_{\nu_e}\gamma_\sigma\gamma_5\hat{p}_e\gamma^\lambda] \tag{H.4}$$

Therefore we need v) of (G.34)

$$Tr[\gamma_\alpha\gamma_\sigma\gamma_\delta\gamma_\lambda] = 4(g_{\alpha\sigma}g_{\delta\lambda} - g_{\alpha\delta}g_{\sigma\lambda} + g_{\alpha\lambda}g_{\sigma\delta}) \tag{H.5}$$

and vii) of (G.35)

$$Tr[\gamma_\alpha\gamma_\sigma\gamma_5\gamma_\delta\gamma_\lambda] = 4i\epsilon_{\alpha\sigma\delta\lambda} \tag{H.6}$$

Therefore

$$\begin{aligned} Tr[\hat{p}_{\nu_e}\gamma_\sigma(1-\gamma_5)\hat{p}_e\gamma^\lambda(1-\gamma_5)] &= 2Tr[\hat{p}_{\nu_e}\gamma_\sigma\hat{p}_e\gamma^\lambda] - 2Tr[\hat{p}_{\nu_e}\gamma_\sigma\gamma_5\hat{p}_e\gamma^\lambda] \\ &= 8p^\alpha_{\nu_e}p^\delta_e{\chi_{\alpha\sigma\delta}}^\lambda \end{aligned} \tag{H.7}$$

where

$$\chi_{\alpha\sigma\delta\lambda} = g_{\alpha\sigma}g_{\delta\lambda} - g_{\alpha\delta}g_{\sigma\lambda} + g_{\alpha\lambda}g_{\sigma\delta} + i\epsilon_{\alpha\sigma\delta\lambda} \tag{H.8}$$

Furthermore we have

$$\begin{aligned} Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)(\hat{p}_\mu+m_\mu)\gamma^\sigma(1-\gamma_5)] &= Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)\hat{p}_\mu\gamma^\sigma(1-\gamma_5)] \\ &+ Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)m_\mu\gamma^\sigma(1-\gamma_5)] \\ &= Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)\hat{p}_\mu\gamma^\sigma(1-\gamma_5)] \\ &\quad +Tr[\hat{p}_{\nu_\mu}\gamma_\lambda m_\mu\gamma^\sigma] - Tr[\hat{p}_{\nu_\mu}\gamma_\lambda\gamma_5 m_\mu\gamma^\sigma] \\ &\quad -Tr[\hat{p}_{\nu_\mu}\gamma_\lambda m_\mu\gamma^\sigma\gamma_5] + Tr[\hat{p}_{\nu_\mu}\gamma_\lambda\gamma_5 m_\mu\gamma^\sigma\gamma_5] \\ &= Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)\hat{p}_\mu\gamma^\sigma(1-\gamma_5)] \\ &= p^\tau_{\nu_\mu}p^\rho_\mu{\chi_{\tau\lambda\rho}}^\sigma \end{aligned} \tag{H.9}$$

where we have used

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho] = 0 \tag{H.10}$$

and the properties of the γ_5 matrix. Using

$${\chi_{\alpha\sigma\delta}}^\lambda{\chi_{\tau\lambda\rho}}^\sigma = 4g_{\alpha\rho}g_{\sigma\tau} \tag{H.11}$$

and substituting in (H.3), we obtain

$$\begin{aligned} \frac{1}{2}\sum_{r_i,r_f}|\mathcal{M}_{fi}|^2 &= \frac{1}{2}\frac{1}{16m_\mu m_{\nu_e}m_{\nu_\mu}m_e}Tr[\hat{p}_{\nu_e}\gamma_\sigma(1-\gamma_5)\hat{p}_e\gamma^\lambda(1-\gamma_5)] \\ &\quad Tr[\hat{p}_{\nu_\mu}\gamma_\lambda(1-\gamma_5)(\hat{p}_\mu+m_\mu)\gamma^\sigma(1-\gamma_5)] \\ &= 8\frac{1}{m_{\nu_\mu}m_{\nu_e}m_\mu m_e}(p_\mu\cdot p_{\bar{\nu}_e})(p_{\nu_\mu}\cdot p_e) \end{aligned} \tag{H.12}$$

I Bose Einstein and Fermi Dirac statistics

Let us now review the quantum statistics. We know from Quantum Mechanics that there are two types of particles, bosons and fermions. Single states can be occupied by any number of bosons while for fermions a single state can be occupied at most by one fermion.

Since atoms are composed of spin 1/2 particles (neutrons, protons and electrons) there are atoms which are bosons (H^1, He^4) and atoms which are fermions (H^2, He^3). Let us now compute the gran partition function for free bosons and fermions.Thermodynamic quantities are derived by the gran partition function $\mathcal{Z}$, since $\mathcal{Z}$ is connected with the thermodynamic potential via

$$\Omega = -k_B T \log \mathcal{Z} \tag{I.1}$$

and the average number of particles and the gas pressure are given by:

$$N = -\frac{\partial \Omega}{\partial \mu}, \quad p = -\frac{\partial \Omega}{\partial V} \tag{I.2}$$

Let H be the Hamiltonian for N free particles

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} \tag{I.3}$$

Let us suppose that for every momentum $\mathbf{p}$ there are $n_\mathbf{p}$ particles with such momentum. Since we are working in a box, $\mathbf{p} = 2\pi/L\mathbf{m}$ with $\mathbf{m} = (m_x, m_y, m_z)$ integers. Therefore we have

$$E = \sum_\mathbf{p} n_\mathbf{p} \epsilon_\mathbf{p} \equiv E(n_\mathbf{p}) \quad N = \sum_\mathbf{p} n_\mathbf{p} \tag{I.4}$$

The gran partition function is given by

$$\mathcal{Z}(\mu, V, T) = \sum_N \sum_{\{n_\mathbf{p}\}} g(n_\mathbf{p}) \exp(-\beta E(n_\mathbf{p}) + \beta \mu N) \tag{I.5}$$

with $g(n_{\mathbf{p}}) = 1$ since all particles are identical. Fer fermions $n_{\mathbf{p}} = 0, 1$ while for bosons $n_{\mathbf{p}} = 0, 1, 2, \cdots$. So we have

$$\begin{aligned}
\mathcal{Z}(\mu, V, T) &= \sum_{N=0}^{\infty} \sum_{\{n_{\mathbf{p}}\}} \exp(-\beta E(n_{\mathbf{p}}) + \beta\mu N) \\
&= \sum_{N=0}^{\infty} \sum_{\{n_{\mathbf{p}}\}} \exp[-\beta \sum_{\mathbf{p}} (n_{\mathbf{p}}\epsilon_{\mathbf{p}} - \mu n_{\mathbf{p}})] \\
&= \sum_{N=0}^{\infty} \sum_{\{n_{\mathbf{p}}\}} \Pi_{\mathbf{p}} \left[\exp(\beta(\mu - \epsilon_{\mathbf{p}}))\right]^{n_{\mathbf{p}}} \\
&= \sum_{n_0} \sum_{n_1} \left[\exp(\beta(\mu - \epsilon_0))\right]^{n_0} \left[\exp(\beta(\mu - \epsilon_1))\right]^{n_1} \cdots \\
&= \Pi_{\mathbf{p}} \sum_{n_{\mathbf{p}}} \left[\exp(\beta(\mu - \epsilon_{\mathbf{p}}))\right]^{n_{\mathbf{p}}} \\
&= \Pi_{\mathbf{p}} \mathcal{Z}_{\mathbf{p}} \qquad \text{(I.6)}
\end{aligned}$$

Let us now consider a gas of fermions, then

$$\mathcal{Z}_{\mathbf{p}}^F = \sum_{n_{\mathbf{p}}=0,1} \left[\exp(\beta(\mu - \epsilon_{\mathbf{p}}))\right]^{n_{\mathbf{p}}} = 1 + \exp[\beta(\mu - \epsilon_{\mathbf{p}})] \qquad \text{(I.7)}$$

For a boson gas we have

$$\mathcal{Z}_{\mathbf{p}}^B = \sum_{n_{\mathbf{p}}=0,1,2,\ldots} \left[\exp(\beta(\mu - \epsilon_{\mathbf{p}}))\right]^{n_{\mathbf{p}}} = \frac{1}{1 - \exp[\beta(\mu - \epsilon_{\mathbf{p}})]} \qquad \text{(I.8)}$$

Note that in the boson case the series converges only if

$$\exp \beta(\mu - \epsilon_{\mathbf{p}}) < 1 \qquad \text{(I.9)}$$

Therefore if the ground level is for $\epsilon_0 = 0$ the chemical potential must be negative. Finally we can calculate the thermodynamic potential:

$$\begin{aligned}
\Omega^F &= -k_B T \sum_{\mathbf{p}} \ln[1 + \exp(\beta(\mu - \epsilon_{\mathbf{p}}))] \\
\Omega^B &= k_B T \sum_{\mathbf{p}} \ln[1 - \exp(\beta(\mu - \epsilon_{\mathbf{p}}))] \qquad \text{(I.10)}
\end{aligned}$$

Given the energy $\epsilon_{\mathbf{p}}$ we can calculate all the thermodynamic quantities. Let us first compute the average number:

$$\begin{aligned} N^F &= -\frac{\partial \Omega^F}{\partial \mu} \equiv \sum_{\mathbf{p}} < n_{\mathbf{p}} >= \sum_{\mathbf{p}} \frac{1}{\exp(\beta(\epsilon_{\mathbf{p}} - \mu)) + 1} \\ N^B &= -\frac{\partial \Omega^B}{\partial \mu} \equiv \sum_{\mathbf{p}} < n_{\mathbf{p}} >= \sum_{\mathbf{p}} \frac{1}{\exp(\beta(\epsilon_{\mathbf{p}} - \mu)) - 1} \end{aligned} \tag{I.11}$$

At low temperature bosons tend to accumulate in the ground state ($\mathbf{p} = 0$), only thermal fluctuations can invert the process. In fact N^B increases when $\epsilon_{\mathbf{p}} - \mu \to 0$.

The classical limit, the Boltzmann distribution, is obtained for $\exp(\beta(\epsilon_{\mathbf{p}} - \mu)) >> 1$ or $\exp(\beta(\mu - \epsilon_{\mathbf{p}})) << 1$. This corresponds to $\exp(\mu/k_B T) << \exp(\epsilon_{\mathbf{p}}/k_B T)$ or large T and $\mu/k_B T \to -\infty$.

Let us now compute the fermion partition function Ω^F for a free particle gas by going in the continuum ($V \to \infty$):

$$\Omega^F = -k_B T \frac{4\pi V}{h^3} \int_0^\infty p^2 dp \ln[1 + \exp((\beta(\mu - \frac{p^2}{2m}))] \tag{I.12}$$

We can now derive average pressure and number of fermions as

$$p = -\frac{\partial \Omega^F}{\partial V} = k_B T \frac{4\pi}{h^3} \int_0^\infty p^2 dp \ln[1 + \exp(\beta(\mu - \frac{p^2}{2m}))] \tag{I.13}$$

$$N = \frac{4\pi V}{h^3} \int_0^\infty p^2 dp \frac{1}{1 + \exp(\beta(-\mu + \frac{p^2}{2m}))} \tag{I.14}$$

All the results can be expressed in terms of the functions

$$\begin{aligned} f_{3/2}(z) &= z \frac{d}{dz} f_{5/2}(z) = \sum_{l=1}^\infty (-)^{l+1} \frac{z^l}{l^{3/2}} \\ f_{5/2}(z) &= \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \ln(1 + z \exp(-x^2)) = \sum_{l=1}^\infty (-)^{l+1} \frac{z^l}{l^{5/2}} \end{aligned} \tag{I.15}$$

where $z = \exp(\beta\mu)$:

$$\begin{aligned} \frac{p}{k_B T} &= \frac{1}{\lambda^3} f_{5/2}(z) \\ \frac{N}{V} &= \frac{1}{\lambda^3} f_{3/2}(z) \end{aligned} \tag{I.16}$$

where

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}} \tag{I.17}$$

For a Bose gas the partition function is singular when $\mathbf{p} = 0$ and $\mu \to 0$ or $z \to 1$. Therefore it is convenient to separate in the sum the term with $\mathbf{p} = 0$ before passing to the continuum.

We get

$$p = -k_BT\frac{4\pi}{h^3}\int_0^\infty p^2 dp \ln[1-\exp(\beta(\mu-\frac{p^2}{2m}))] - \frac{k_BT}{V}\ln[1-\exp(\beta\mu))] \tag{I.18}$$

$$N = V\frac{4\pi}{h^3}\int_0^\infty p^2 dp \frac{1}{-1+\exp(\beta(-\mu+\frac{p^2}{2m}))} + N_0 \tag{I.19}$$

where

$$N_0 = \frac{\exp(\beta\mu)}{1-\exp(\beta\mu)} \equiv \frac{z}{1-z} \tag{I.20}$$

N_0 denotes the number of particles in the $\mathbf{p} = 0$ state.

The results can now be written in terms of the functions

$$\begin{aligned} g_{3/2}(z) &= z\frac{d}{dz}g_{5/2}(z) = \sum_{l=1}^{\infty}\frac{z^l}{l^{3/2}} \\ g_{5/2}(z) &= -\frac{4}{\sqrt{\pi}}\int_0^\infty dx x^2 \ln(1-z\exp(-x^2)) = \sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}} \end{aligned} \tag{I.21}$$

We have

$$\begin{aligned} \frac{p}{k_BT} &= \frac{1}{\lambda^3}g_{5/2}(z) - \frac{1}{V}\ln(1-z) \\ \frac{N}{V} &= \frac{1}{\lambda^3}g_{3/2}(z) + \frac{N_0}{V} \end{aligned} \tag{I.22}$$

I.1 A gas of free fermions

Let us now rewrite the equations (I.16) for the pressure and concentration for a gas of fermions:

$$\begin{aligned} \frac{p}{k_BT} &= \frac{1}{\lambda^3}f_{5/2}(z) \\ \frac{N}{V} &= \frac{1}{\lambda^3}f_{3/2}(z) \end{aligned} \tag{I.23}$$

The equation of state is obtained by eliminating z from the two equations. Let us start with the equation

$$\frac{\lambda^3}{v} = f_{3/2}(z) \tag{I.24}$$

Therefore it is convenient to study the function $f_{3/2}$ as a function of z. $f_{3/2}$ is a function monotone in z. For small z

$$f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} - \frac{z^4}{4^{3/2}} + \cdots \tag{I.25}$$

For large z (Huang p.246)

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}}[(\ln z)^{3/2} + \frac{\pi^2}{8}\frac{1}{(\ln z)^{1/2}}] + O(1/z) \tag{I.26}$$

Therefore for every positive value of λ a solution for z exists.

Low density and high temperature, $\lambda^3/v << 1$

In this case the thermal length $\lambda \sim \hbar/p \sim \hbar/\sqrt{mk_BT}$ is much smaller than the average distance among the particles $v^{1/3}$, therefore quantum effects are negligible. From

$$\frac{\lambda^3}{v} = z - \frac{z^2}{2^{3/2}} + \ldots \tag{I.27}$$

one gets

$$z = \frac{\lambda^3}{v} + \frac{1}{2^{3/2}}(\frac{\lambda^3}{v})^2 + \ldots) \tag{I.28}$$

and the equation of state becomes

$$\frac{pV}{k_BTN} = \frac{v}{\lambda^3}(z - \frac{z^2}{2^{5/2}} + \ldots = 1 + \frac{1}{2^{5/2}}\frac{\lambda^3}{v} + \ldots \tag{I.29}$$

Therefore one obtains quantum corrections to the classic case. Recalling the virial expansion

$$\frac{p}{k_BT} = \frac{N}{V}(1 + B(T)\frac{N}{V} + C(T)\left(\frac{N}{V}\right)^2 + \ldots) \tag{I.30}$$

one can identify the virial coefficients.

High density and low temperature, $\lambda^3/v >> 1$

In this case the thermal distance is much larger than the average distance so the quantum effects become relevant. The leading term is now

$$\frac{\lambda^3}{v} = \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2} \tag{I.31}$$

Therefore we get

$$z = \exp\left(\beta\epsilon_F\right) \tag{I.32}$$

where the chemical potential ϵ_F is called *Fermi energy*

$$\epsilon_F = \frac{\hbar^2}{2m}\left[\frac{6\pi^2}{v}\right]^{2/3} \tag{I.33}$$

Let us now consider $< n_{\mathbf{p}} >$, defined in eqs.(I.11)

$$< n_{\mathbf{p}} >= \frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\epsilon_F)}+1} \tag{I.34}$$

When $T \to 0$, $\beta \to +\infty$ we have

$$< n_{\mathbf{p}} >_{T=0} = 1 \tag{I.35}$$

for $\epsilon_{\mathbf{p}} < \epsilon_F$ and

$$< n_{\mathbf{p}} >_{T=0} = 0 \tag{I.36}$$

for $\epsilon_{\mathbf{p}} > \epsilon_F$.

Therefore at zero temperature the fermions occupy all the lowest levels up to ϵ_F. Because of the Pauli principle they cannot occupy all the ground state and therefore they fill all the states up to the highest energy ϵ_F. Such a state is called a *degenerate Fermi gas*. In the momentum space the particles fill a sphere of radius p_F, the *Fermi surface*. One defines also a *Fermi temperature* or *degeneracy temperature* T_F such that

$$k_B T_F = \epsilon_F \tag{I.37}$$

Finally we can compute the internal energy

$$U = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} n_{\mathbf{p}} = \frac{V}{h^3}\frac{4\pi}{2m}\int_0^\infty dp p^4 < n_{\mathbf{p}} > \tag{I.38}$$

By part integration we get

$$U = \frac{V}{4\pi^2 m\hbar^3}\int_0^\infty dp\frac{p^5}{5}\left(-\frac{\partial}{\partial p}n_{\mathbf{p}}\right) = \frac{\beta V}{20\pi^2 m^2\hbar^3}\int_0^\infty dp\frac{p^6 e^{\beta\epsilon_{\mathbf{p}}-\beta\mu}}{(e^{\beta\epsilon_{\mathbf{p}}-\beta\mu}+1)^2} \tag{I.39}$$

The integrand has a peak at $p = p_F$. The asymptotic behavior of the integral is (see [?])

$$U = \frac{3}{5}N\epsilon_F\left[1+\frac{5}{12}\pi^2(\frac{k_BT}{\epsilon_F})^2\right] \tag{I.40}$$

From the internal energy one can derive the specific heat

$$C_V = Nk_B\frac{\pi^2}{2}\frac{k_BT}{\epsilon_F} \tag{I.41}$$

which goes to zero when $T \to 0$ (Third law of thermodynamics) and the pressure

$$p = \frac{2}{3}\frac{U}{V} = \frac{2}{5}\frac{\epsilon_F}{v}\left[1+\frac{5}{12}\pi^2(\frac{k_BT}{\epsilon_F})^2\right] \tag{I.42}$$

Notice that even at $T = 0$ as a consequence of Pauli principle the gas has a non vanishing pressure. This pressure is responsible for the gravitational stability of white dwarfs and neutron stars. A white dwarf can be thought as a gas of ionized helium and electrons. The gravitational stability is guaranteed by the degenerate electron pressure. In the case of neutron stars the stability is guaranteed by the pressure of the degenerate gas of neutrons.

Note Alternative way to compute ϵ_F.

An alternative way of computing ϵ_F is to fill all the states in the momentum space up to p_F:

$$N = \frac{4\pi V}{h^3}\int_0^{p_F} p^2 dp = \frac{4\pi V}{3h^3}p_F^3 = \frac{V}{6\pi^2\hbar^3}p_F^3 \tag{I.43}$$

or

$$p_F = (\frac{N}{V})^{1/3}(6\pi^2)^{1/3}\hbar \tag{I.44}$$

$$\epsilon_F = \frac{1}{2m}(\frac{N}{V})^{2/3}3^{2/3}h^2 = \frac{\hbar^2}{2m}\left[\frac{6\pi^2}{v}\right]^{2/3} \tag{I.45}$$

I.2 A gas of free bosons

Let us now study with some detail the Bose case. The function $g_{3/2}$ for z small can be studied as a series

$$g_{3/2}(z) = z + \frac{1}{2^{3/2}} z^2 + \frac{1}{3^{3/2}} z^3 + \cdots \tag{I.46}$$

with

$$g_{3/2}(1) = \sum_{l=1}^{\infty} \frac{1}{l^{3/2}} = \zeta(\frac{3}{2}) = 2.612... \tag{I.47}$$

where ζ is the Riemann function. As we have already noticed for a boson gas $\mu < 0$ then $0 \leq z \leq 1$ and $g_{3/2} \leq 1$. Rewriting the eq.(I.22) for the average number

$$\lambda^3 \frac{N_0}{V} = \frac{\lambda^3}{v} - g_{3/2}(z) \tag{I.48}$$

with

$$v = \frac{V}{N} \tag{I.49}$$

we see that $N_0/V > 0$ if temperature and specific volume v are such that

$$\frac{\lambda^3}{v} > g_{3/2}(1) \tag{I.50}$$

In fact

$$0 < \frac{\lambda^3}{v} - g_{3/2}(1) < \frac{\lambda^3}{v} - g_{3/2}(z) \tag{I.51}$$

This means that the ground state is occupied by a macroscopic fraction of bosons. The critical temperature for the Bose condensation is defined by

$$\frac{\lambda_c^3}{v} = \frac{1}{v} \left(\frac{2\pi\hbar^2}{m k_B T_c} \right)^{3/2} = g_{3/2}(1), \quad or \;\; \mu = 0, \;\; N_0 = 0 \tag{I.52}$$

or

$$T_c = \frac{1}{k_B} \frac{2\pi\hbar^2/m}{[v g_{3/2}(1)]^{2/3}} = \frac{1}{k_B} \frac{2\pi\hbar^2/m}{[\zeta(3/2)]^{2/3}} (\frac{N}{V})^{2/3} \tag{I.53}$$

At the critical temperature the bosons start to occupy the $\mathbf{p} = 0$ state and if the temperature decreases more and more bosons occupy such a state. For $T < T_c$ the chemical potential μ remain zero. Inserting the values $\rho_{He4} = 0.145 \mathrm{g/cm}^3 \sim m_{He^4} N/V$ with $m_{He^4} = 4 m_p$, $m_p = 4 \times 1.67 \times 10^{-27}$Kg, $\hbar = 1.05510^{-34}$ J s, the Boltzmann constant $k = 1.38\ 10^{-23}$J/°K) we get $T_c \sim 3.14$ °K. This temperature is very close to the critical temperature of liquid Helium, $T_\lambda = 2.17$ °K, below which the helium becomes superfluid.

J Fundamental state of the superfluidity theory

Let us now discuss the properties of the new vacuum state $|\tilde{\phi}_0 >$ of the Hilbert space of quantum states of superfluidity. The usual form of quantum field theory vacuum cannot be used since the fundamental state for a system of N bosons is given by

$$|\phi_0(N) >= |N, 0, \cdots 0 > \tag{J.54}$$

that means that all the particles are in the lowest energy state ($k = 0$). Therefore the annihilation operator a_0 does not annihilate the minimum energy state

$$a_0|0 >= 0 \tag{J.55}$$

but

$$a_0|\phi_0(N) >= N^{1/2}|\phi_0(N-1) > \tag{J.56}$$

e

$$a_0^\dagger|\phi_0(N) >= (N+1)^{1/2}|\phi_0(N+1) > \tag{J.57}$$

To find the minimum energy state, it is necessary to consider first the coherent state

$$|\phi_0 >= A^{1/2}\exp[\sqrt{V}\phi_0 a_0^\dagger]|0 > \tag{J.58}$$

which satisfies

$$a_0|\phi_0 >= \sqrt{V}\phi_0|\phi_0 > \tag{J.59}$$

and

$$a_{\mathbf{k}}|\phi_0 >= 0 \;\; \mathbf{k} \neq 0 \tag{J.60}$$

$$n_0 = \frac{< N(k=0) >}{V} = \frac{1}{V} < \phi_0|a_0^\dagger a_0|\phi_0 >= \frac{1}{V}V\phi_0^2 \tag{J.61}$$

In other words the expectation value of N is $V\phi_0^2$. The normalization is given by

$$A^{1/2} = \exp[-\frac{1}{4}V\phi_0^2] \tag{J.62}$$

Therefore n_0 is the boson density in the state $k = 0$. The vacuum expectation value of the field $\phi(x)$ on the state $|\phi_0 >$ is given by

$$< \phi_0|\phi(x)|\phi_0 >= \phi_0 = \sqrt{n_0} = \sqrt{\frac{< N(k=0) >}{V}} \tag{J.63}$$

and it is related to the density of the condensate. The true vacuum state is however defined as we have found in Section 11.4 by the condition (11.54)

$$A_{\mathbf{k}}|\tilde{\phi}_0>= \left[\cosh(\frac{\theta_k}{2})a_{\mathbf{k}} + \sinh(\frac{\theta_k}{2})a^\dagger_{-\mathbf{k}}\right]|\tilde{\phi}_0>= 0 \tag{J.64}$$

The solution is given by

$$|\tilde{\phi}_0>= \tilde{N}\exp\,[-\frac{1}{2}\sum_{k\neq 0}\tanh(\theta_k/2)a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}]|\phi_0> \tag{J.65}$$

This means that the true vacuum state contains pair of bosons with opposite momenta.

Exercise. Verify the at eq. (J.64) is satisfied by the new vacuum (J.65).

Exercise. Verify that the state $|\tilde{\phi}_0>$ corresponds to a lower value of the energy with respect to $|\phi_0>$.

K Bogoliubov transformation

Let us now derive the Bogoliubov transformation. Let us start considering

$$\sum_{k\neq 0}\left[\alpha_k a^\dagger_k a_k + \frac{\mu}{2}(a_k a_{-k} + a^\dagger_k a^\dagger_{-k})\right] \tag{K.1}$$

where

$$\alpha_k = \mu + \frac{\hbar^2 k^2}{2m} \tag{K.2}$$

Let us consider

$$A_k = \beta_k a_k + \gamma_k a^\dagger_{-k} \tag{K.3}$$

with $\beta_k, \gamma_k \in \mathbb{R}$. Then we get

$$[A_k, A^\dagger_{k'}] = [\beta_k a_k + \gamma_k a^\dagger_{-k}, \beta_k a^\dagger_{k'} + \gamma_k a_{-k'}] = (\beta_k^2 - \gamma_k^2)\delta_{kk'} \tag{K.4}$$

In order to get standard commutation relations, let us require

$$\beta_k^2 - \gamma_k^2 = 1 \tag{K.5}$$

It is convenient to define

$$\beta_k = \cosh\left(\frac{\theta_k}{2}\right), \quad \gamma_k = \sinh\left(\frac{\theta_k}{2}\right) \tag{K.6}$$

The inverse transformations are

$$a_k = \beta_k A_k - \gamma_k A^\dagger_{-k}, \quad a^\dagger_k = \beta_k A^\dagger_k - \gamma_k A_{-k} \tag{K.7}$$

In fact

$$\beta_k A_k - \gamma_k A^\dagger_{-k} = \beta_k(\beta_k a_k + \gamma_k a^\dagger_{-k}) - \gamma_k(\beta_k a^\dagger_{-k} + \gamma_k a_k) = a_k \tag{K.8}$$

Substituting in eq.(K.1) one obtains

$$\begin{aligned}
\sum_{k\neq 0}\left[\alpha_k a^\dagger_k a_k + \frac{\mu}{2}(a_k a_{-k} + a^\dagger_k a^\dagger_{-k})\right] &= \sum_{k\neq 0}\Big[\alpha_k(\beta_k A^\dagger_k - \gamma_k A_{-k})(\beta_k A_k - \gamma_k A^\dagger_{-k}) \\
&\quad + \frac{\mu}{2}\Big((\beta_k A_k - \gamma_k A^\dagger_{-k})(\beta_k A_{-k} - \gamma_k A^\dagger_k) \\
&\quad + (\beta_k A^\dagger_k - \gamma_k A_{-k})(\beta_k A^\dagger_{-k} - \gamma_k A_k)\Big)\Big] \\
&= \sum_{k\neq 0}\Big[\Big((\beta_k^2 + \gamma_k^2)\alpha_k - 2\beta_k\gamma_k\mu\Big)A^\dagger_k A_k + \\
&\quad \Big(-\beta_k\gamma_k\alpha_k + \frac{\mu}{2}(\beta_k^2 + \gamma_k^2)\Big)(A_k A_{-k} + A^\dagger_k A^\dagger_{-k}) \\
&\quad + \alpha_k\gamma_k^2 - \beta_k\gamma_k\mu\Big]
\end{aligned} \tag{K.9}$$

By requiring the vanishing of the coefficient of $A_k A_{-k} + A^\dagger_k A^\dagger_{-k}$ we get

$$\tanh\theta_k = \frac{2\beta_k\gamma_k}{\beta_k^2 + \gamma_k^2} = \frac{\mu}{\alpha_k} \tag{K.10}$$

Then the coefficient of $A^\dagger_k A_k$ becomes, using (K.10) and (K.6)

$$\begin{aligned}
(\beta_k^2 + \gamma_k^2)\alpha_k - 2\beta_k\gamma_k\mu &= (\beta_k^2 + \gamma_k^2)\alpha_k - \frac{4\beta_k^2\gamma_k^2\alpha_k}{\beta_k^2 + \gamma_k^2} = \frac{(\beta_k^2 - \gamma_k^2)^2\alpha_k}{\beta_k^2 + \gamma_k^2} = \frac{\alpha_k}{\beta_k^2 + \gamma_k^2} \\
&= \frac{\alpha_k}{\cosh\theta_k} = \alpha_k\sqrt{1 - \tanh^2\theta_k} = \sqrt{\alpha_k^2 - \mu^2} \\
&\equiv \epsilon(k)
\end{aligned} \tag{K.11}$$

with $\epsilon(k)$ given by eq.(11.56). Finally

$$\alpha_k\gamma_k^2 - \beta_k\gamma_k\mu = \alpha_k\gamma_k^2\frac{\gamma_k^2 - \beta_k^2}{\beta_k^2 + \gamma_k^2} = -\frac{\alpha_k\gamma_k^2}{\beta_k^2 + \gamma_k^2} = -\epsilon(k)\sinh^2\left(\frac{\theta_k}{2}\right) \tag{K.12}$$

where use has been made of eq.(K.11).

References

[1] E.T.Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge Univ. Press, 1904.

[2] J.D. Bjorken, S. D. Drell, Relativistic Quantum Mechanics, Relativistic Quantum Fields, McGraw-Hill, 1965

[3] S. Weinberg, The Quantum Theory od Fields, Cambridge Univ. Press, 1993

[4] K. A. Milton, The Casimir Effect: Physical Manifestations of Zero-point Energy, World Scientific, 2001

[5] S. S. Schweber, An Introduction to Relativistic Quantum Filed Theory, Harper and Row, 1964

[6] J. D. Jackson, Classical Electrodynamics, Wiley, 1998

[7] W. Greiner, Quantum Mechanics, Special Chapters, Springer 1998

[8] K. Huang, Statistical Mechanics, Wiley, 1987

[9] E.M. Lifshitz and L.P.Pitaevskii, Landau and Lifshitz, Course of Theoretical Physics, Statistical Physics, part 2, Pergamon Press,

[10] A.L. Fetter and J.D. Walecka, Quantum Theory of Many-Particle Systems, McGraw-Hill, 1971

[11] D. J. Amit and Y. Verbin, Statistical Physics, An Introductory Course, World Scientific, 1999

[12] D.R. Tilley and J. Tilley, Superconductivity and Superfluidity, Adam Hilber LTD, Bristol, 1986

[13] S. J. Chang, Introduction to Quantum Field Theory, World Scientific, Singapore 1990.

[14] F. Mandl and G. Shaw, Quantum Field Theory, John Wiley and Sons, 1984

[15] R. Casalbuoni, Introduction to Quantum Field Theory, World Scientific Publishing, Singapore, 2011

[16] H. R. Glyde, Excitations in Liquid and Solid Helium, Clarendon Press, Oxford, 1994

[17] F. Gross, Relativistic Quantum Mechanics and Field Theory, John Wiley and sons, 1993

[18] Particle Data Group, http://pdg.lbl.gov/

[19] CMS collaboration, arXiv:2009.04363

[20] J.J. Sakurai, Advanced Quantum Mechanics, Addison Wesley pub. Company, 1967

[21] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, Photons & Atoms, Introduction to Quantum Electrodynamics, John Wiley and Sons New York, 1989

[22] W. Rudin, Functional Analysis, Mc Graw-Hill, New York, 1991

[23] A. Altland and B. Simons, Condensed Matter Field Theory, Cambridge University Press, 2010

[24] Wu Ki Tung, Group Theory in Physics, World Scientific, 1985

[25] Ashcroft N. W. and Mermin N., Solid State Physics, Holt, Rinehart and Winston, 1976